Minimum Rates of Approximate Sufficient Statistics

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Outline

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1. Sufficient Statistics, Motivation, and Main Contribution
2. Problem Setup
3. Main Result and Interpretation

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$$P_{X|\theta}(x) = \sum_{y \in \mathcal{Y}} P_{X|Y}(x|y) P_{Y|\theta}(y) = \sum_{y \in \mathcal{Y}} P_{X,Y|\theta}(x,y) \quad \forall (x,\theta)$$

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- In information theory language,

$$I(\theta; X) = I(\theta; f(X)) = I(\theta; Y).$$

Y provides as much information about θ as X does.

Examples

- $X^n = (X_1, \ldots, X_n) \in \{0,1\}^n$ is i.i.d. Bernoulli with parameter $\theta = \Pr[X_i = 1]$. Then

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- Exponential family with natural parameter $\theta = (\theta_1, \ldots, \theta_d)$

$$P^n_{X|\theta}(x^n) = P^n_X(x^n) \exp\left[\langle Y^{(n)}(x^n), \theta\rangle - nA(\theta)\right].$$

Vector of sufficient statistics $Y^{(n)}(x^n) = (Y_1^{(n)}(x^n), \ldots, Y_d^{(n)}(x^n))$ with

$$Y_i^{(n)}(x^n) = \sum_{j=1}^n Y_i(x_j), \quad i = 1, \ldots, d.$$

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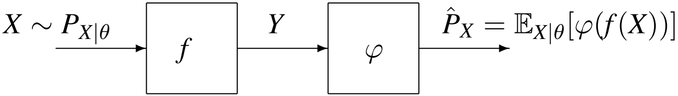

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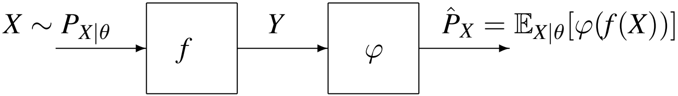

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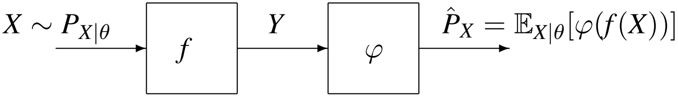

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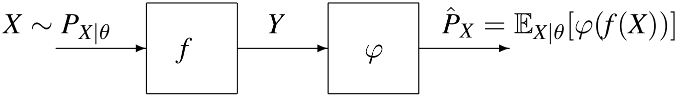

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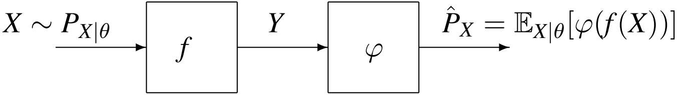

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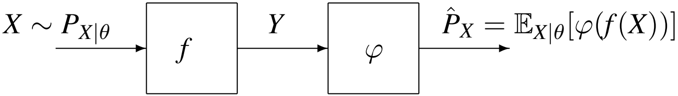

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$$\begin{aligned}\varphi \circ f \circ P_{X|\theta} &= \sum_{y\in\mathcal{Y}} (f \circ P_{X|\theta})(y)\varphi(y) \\ &= \sum_{y\in\mathcal{Y}} P_{X|\theta}\{x \in \mathcal{X} : f(x) = y\} P_{X|Y=y,\theta} = P_{X|\theta}.\end{aligned}$$

Memory Size

- Example 1: Binomial case. Since $\mathcal{X} = \{0, 1\}$, the sufficient statistic

$$\frac{1}{n}\sum_{j=1}^{n} X_j \in \left\{\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\}$$

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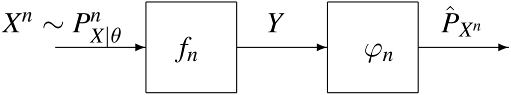
- Example 3: $\theta \in \Theta = [0, 1]$ is the unknown mean of a Gaussian. Sufficient statistics can take uncountable number of values.

Our Contribution

- Reduce d in n^d by relaxing exact recovery condition on $P^n_{X|\theta}$.



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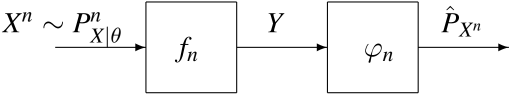

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we only require that

$$\varlimsup_{n\to\infty} \int_\Theta F\Big(\underbrace{\varphi_n \circ f_n \circ P^n_{X|\theta}}_{\text{synthesized}}, \underbrace{P^n_{X|\theta}}_{\text{original}} \Big)\, \mu(\mathrm{d}\theta) \le \delta, \quad \text{for some } \delta \ge 0,$$

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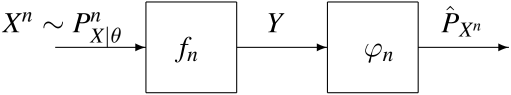

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- Often, we can reduce the exponent to $d/2$ and this is optimal.

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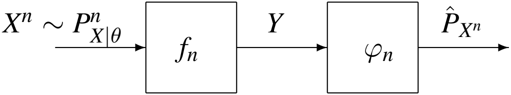

Definition (Code)

A size-M_n code $\mathcal{C}_n = (f_n, \varphi_n)$ consists of

- A possibly stochastic encoder $f_n : \mathcal{X}^n \to \mathcal{Y}_n = \{1, \ldots, M_n\}$;
- A decoder $\varphi_n : \mathcal{Y}_n \to \mathcal{P}(\mathcal{X}^n)$ (set of distributions on $\mathcal{X}^n$)

Definition of Error

Definition (Average Error)

The average error is a code $\mathcal{C}_n = (f_n, \varphi_n)$ is defined as

$$\varepsilon(\mathcal{C}_n) := \int_\Theta F\left(\varphi_n \circ f_n \circ P^n_{X|\theta}, P^n_{X|\theta}\right) \mu(\mathrm{d}\theta)$$
$$= \mathbb{E}_{\theta\sim\mu}\left[F\left(\varphi_n \circ f_n \circ P^n_{X|\theta}, P^n_{X|\theta}\right)\right]$$

where $\mu(\cdot)$ is the prior distribution of θ.

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$$\lim_{n\to\infty} \frac{\log|\mathcal{C}_n|}{\log n} = r \quad \Longleftrightarrow \quad |\mathcal{C}_n| \asymp n^r$$

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- Typically for $\Theta \subset \mathbb{R}^d$,

$$\mathrm{R}^{(i)}(\delta) = \frac{d}{2}.$$

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Theorem (Hayashi and Tan (2018))

1. *Assume (i), (ii), (iv), and (v), under the variational distance criterion*

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3. *If in addition $\{P_{X|\theta}\}_{\theta\in\Theta}$ is an exponential family,*
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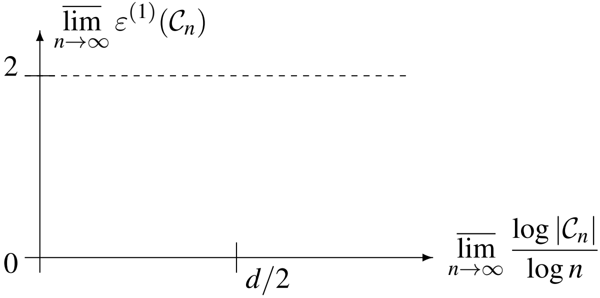
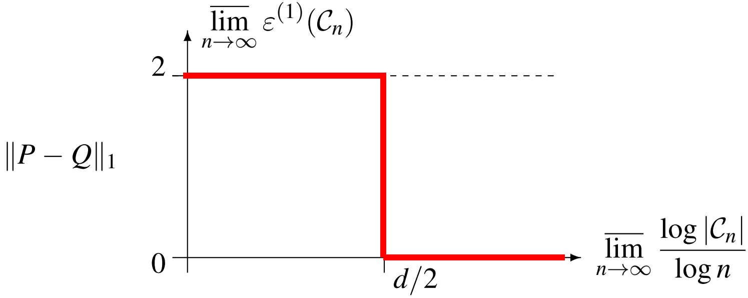
  the memory requirement $d/2$ is asymptotically the same.
- This is known in information theory as a strong converse.

Main Result : Strong Converse

$\|P - Q\|_1$




$\overline{\lim}_{n\to\infty} \varepsilon^{(1)}(\mathcal{C}_n)$
2
$\|P - Q\|_1$
0
$d/2$
$\overline{\lim}_{n\to\infty} \frac{\log|\mathcal{C}_n|}{\log n}$

Outline

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Universal Coding, Information, Prediction, and Estimation

JORMA RISSANEN

Abstract—A connection between universal codes and the problems of prediction and statistical estimation is established. A known lower bound for the mean length of universal codes is sharpened and generalized, and optimum universal codes constructed. The bound is defined to give the information in strings relative to the considered class of processes. The earlier derived minimum description length criterion for estimation of parameters, including their number, is given a fundamental information theoretic justification by showing that its estimators achieve the information in the strings. It is also shown that one cannot do prediction in Gaussian autoregressive moving average (ARMA) processes below a bound, which is determined by the information in the data.

Manuscript received July 13, 1983; revised January 16, 1984. This work was presented in part at the IEEE International Symposium on Information Theory, St. Jovite, Canada, September 26–30, 1983.

This work was done while the author was Visiting Professor at the Department of System Science, University of California, Los Angeles, while on leave from the IBM Research Laboratory, San Jose, CA 95193.

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THERE are three main problems in signal processing: prediction, data compression, and estimation. In the first, we are given a string of observed data points x_t, $t = 1, \cdots, n$, one after another, and the objective is to predict for each t the next outcome x_{t+1} from what we have seen so far. In the data compression problem we are given a similar sequence of observations, each truncated to some finite precision, and the objective is to redescribe the data with a suitably designed code as efficiently as possible, i.e., with a short code length.

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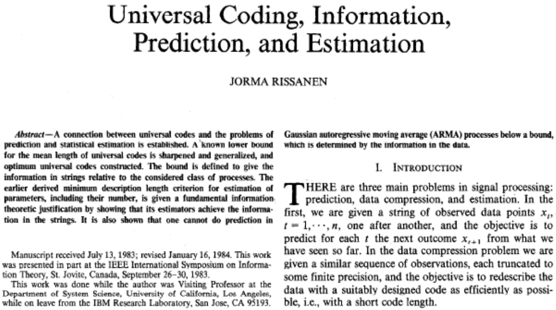
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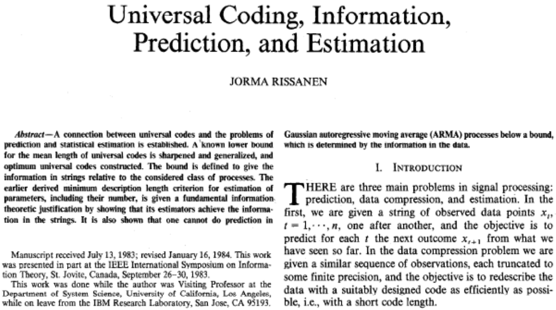

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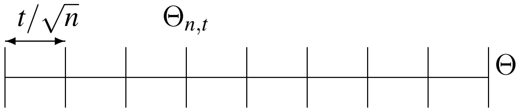

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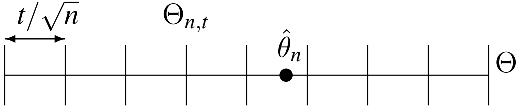

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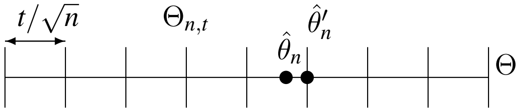

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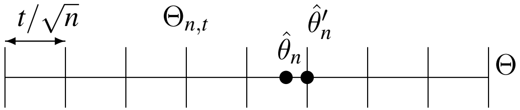

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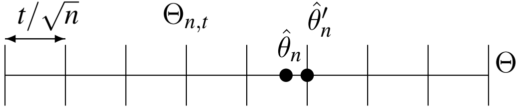

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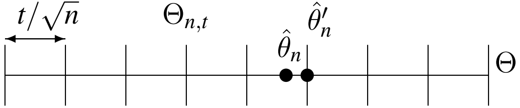

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- Can show that

$$\varlimsup_{n\to\infty} \varepsilon^{(2)}(\mathcal{C}_n) \leq \frac{d}{2}$$

by eventually taking $t \downarrow 0$. But error is non-vanishing. Weak achievability.

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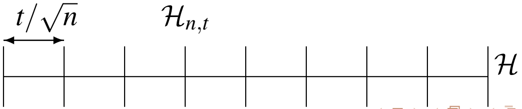

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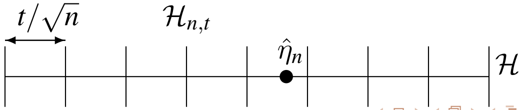

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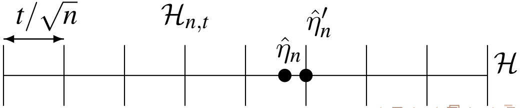

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- Decoder: Uniform mixture of conditional distributions whose moment parameter is discretized to $\hat{\eta}_n'$:

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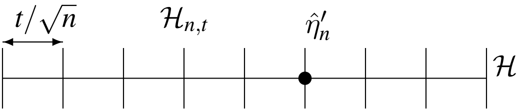

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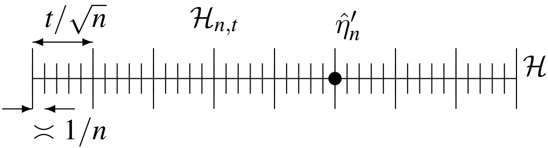

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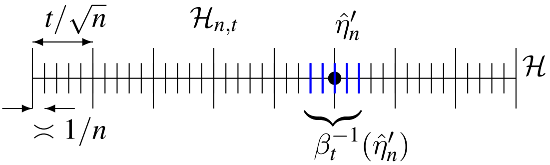

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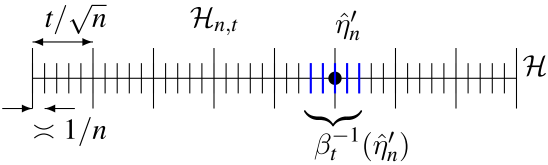

- Asymptotic error under relative entropy is zero and $|\mathcal{H}_{n,t}| \asymp n^{d/2}$.

Outline

Weak Converse Variational Distance: $\mathrm{R}^{(1)}(0) \geq \frac{d}{2}$

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IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 36, NO. 3, MAY 1990 453

Information-Theoretic Asymptotics of Bayes Methods

BERTRAND S. CLARKE AND ANDREW R. BARRON, MEMBER, IEEE

Abstract—**In the absence of knowledge of the true density function, Bayesian models take the joint density function for a sequence of n random variables to be an average of densities with respect to a prior. We examine the relative entropy distance D_n between the true density and the Bayesian density and show that the asymptotic distance is $(d/2)(\log n)+c$, where d is the dimension of the parameter vector. Therefore, the relative entropy rate D_n/n converges to zero at rate $(\log n)/n$. The constant c, which we explicitly identify, depends only on the prior density function and the Fisher information matrix evaluated at the true parameter value. Consequences are given for density estimation, universal data compression, composite hypothesis testing, and stock-market portfolio selection.**

I. Introduction

THE RELATIVE entropy is a mathematical expres-

we identify. We note that if the mixture excludes a neighborhood of the true density, then the behavior of the relative entropy is, asymptotically, of the order of the sample size; in addition, if the prior is discrete and assigns positive mass at θ_o, the relative entropy then asymptotically tends to a constant.

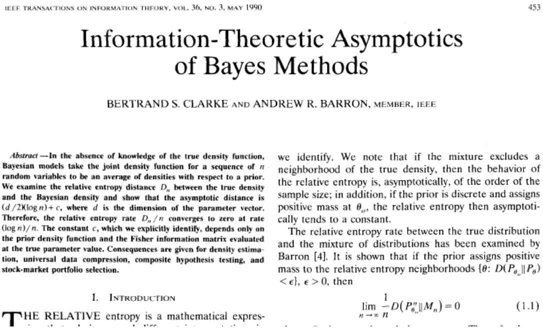
The relative entropy rate between the true distribution and the mixture of distributions has been examined by Barron [4]. It is shown that if the prior assigns positive mass to the relative entropy neighborhoods $\{\theta: D(P_{\theta_o}\|P_\theta) < \epsilon\}$, $\epsilon > 0$, then

$$\lim_{n\to\infty}\frac{1}{n}D(P^n_{\theta_o}\|M_n)=0 \qquad (1.1)$$

B. Clarke

A. Barron

Weak Converse Variational Distance: $\mathrm{R}^{(1)}(0) \geq \frac{d}{2}$



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Abstract —In the absence of knowledge of the true density function, Bayesian models take the joint density function for a sequence of n random variables to be an average of densities with respect to a prior. We examine the relative entropy distance D_n between the true density and the Bayesian density and show that the asymptotic distance is $(d/2)(\log n) + c$, where d is the dimension of the parameter vector. Therefore, the relative entropy rate D_n/n converges to zero at rate $(\log n)/n$. The constant c, which we explicitly identify, depends only on the prior density function and the Fisher information matrix evaluated at the true parameter value. Consequences are given for density estimation, universal data compression, composite hypothesis testing, and stock-market portfolio selection.

I. INTRODUCTION

THE RELATIVE entropy is a mathematical expres-

we identify. We note that if the mixture excludes a neighborhood of the true density, then the behavior of the relative entropy is, asymptotically, of the order of the sample size; in addition, if the prior is discrete and assigns positive mass at θ_o, the relative entropy then asymptotically tends to a constant.

The relative entropy rate between the true distribution and the mixture of distributions has been examined by Barron [4]. It is shown that if the prior assigns positive mass to the relative entropy neighborhoods $\{\theta\colon D(P_{\theta_o} \| P_\theta) < \epsilon\}$, $\epsilon > 0$, then

$$\lim_{n\to\infty} \frac{1}{n} D(P^n_{\theta_o} \| M_n) = 0 \qquad (1.1)$$

B. Clarke

A. Barron

$$D\left(P^n_{X|\theta} \,\middle\|\, \underbrace{\int_\Theta P^n_{X|\theta'}\,\mu(\mathrm{d}\theta')}_{\text{mixture}}\right) = ??$$

Weak Converse Variational Distance: $\mathrm{R}^{(1)}(0) \geq \frac{d}{2}$

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and the uniform continuity of mutual information [Zhang (1997)]:

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- Assume, to the contrary, that $\exists\, \mathcal{C}_n$ and $\alpha \in (0, 2)$ with error

$$\mathbb{E}_{\theta\sim\mu}\left[\left\|P^n_{X|\theta} - (\varphi \circ f)(\theta)\right\|_1\right] \leq 2 - \alpha,$$

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- Define $\mathcal{S} = \left\{\theta\in\Theta : \|P^n_{X|\theta} - (\varphi\circ f)(\theta)\|_1 \leq 2 - \frac{\alpha}{2}\right\}$. Markov inequality says

$$\mu(\mathcal{S}) \geq \frac{\alpha}{4-\alpha} > 0.$$

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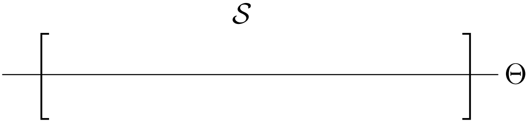

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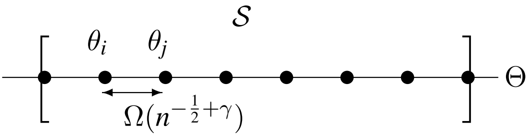

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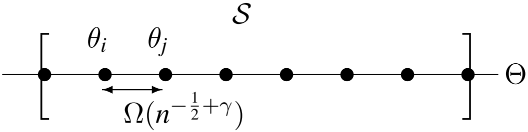

- Because separation is $\Omega(n^{-\frac{1}{2}+\gamma})$, there exists disjoint $\mathcal{D}_i \subset \mathcal{X}^n$, $i = 1, \ldots, \frac{5}{\alpha}M_n$ such that

$$P^n_{X|\theta_i}(\mathcal{D}_i) \geq 1 - \epsilon, \qquad \text{for any } \epsilon \in (0, 1).$$

Follows by weak law of large numbers.

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- We have

$$(\varphi \circ f(\theta_i))(\mathcal{D}_i) \geq \frac{\alpha}{4} - \epsilon, \quad \forall\, i = 1, \ldots, \frac{5}{\alpha} M_n.$$

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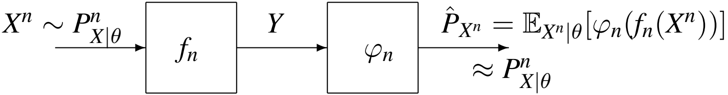
Contradiction if $0 < \epsilon < \frac{\alpha}{20}$.

Outline

Concluding Remarks



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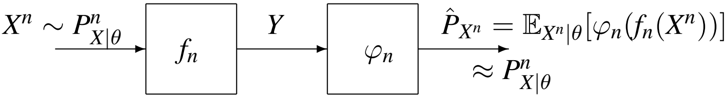

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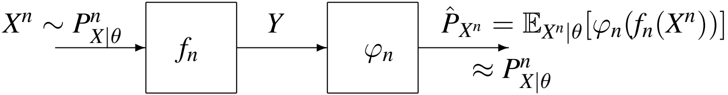

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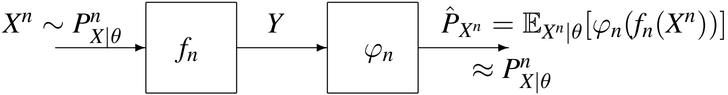

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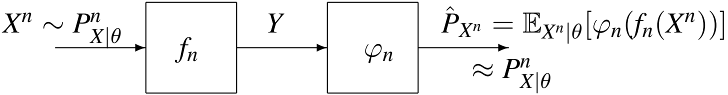

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