How to Measure Side-Channel Leakage

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Packet-Timing Side Channel

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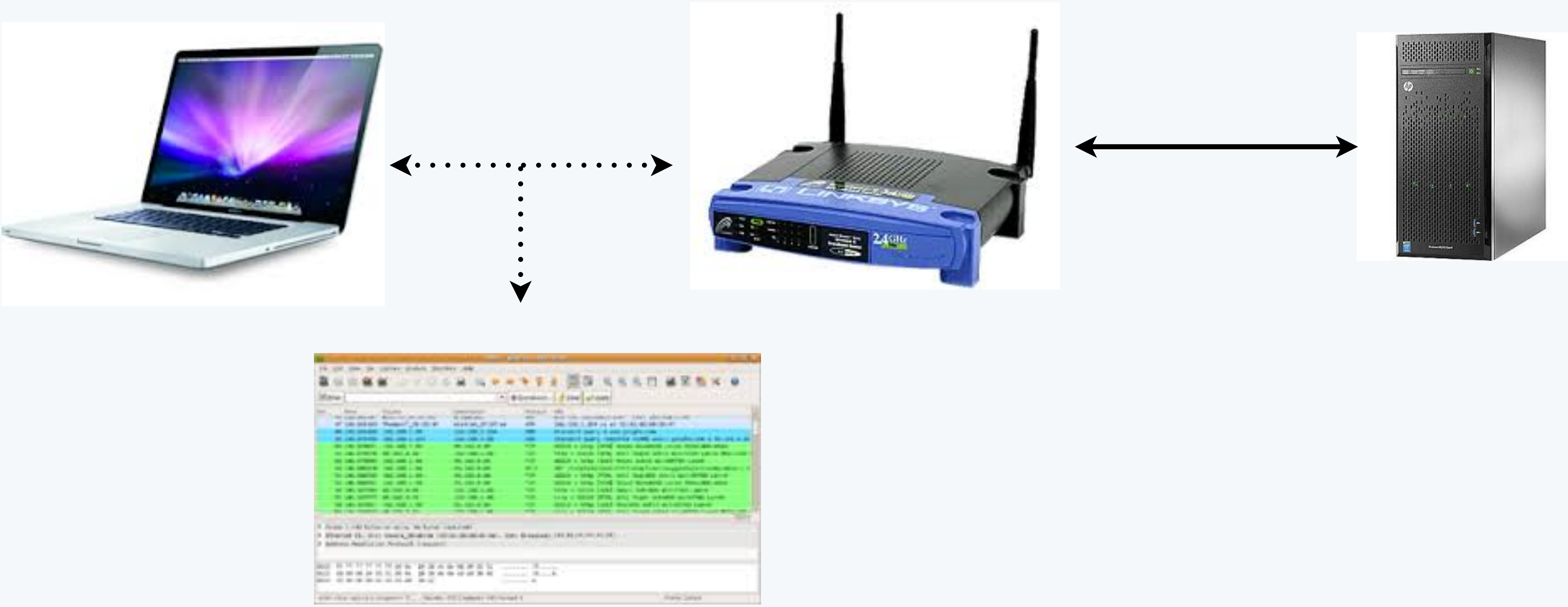

- ssh: keystrokes are sent as separate packets.
- Packet timing ↔ keystroke timing ↔ typed letters
- Packet-sniffing eavesdropper can acquire information about typed characters (e.g. passwords).

[Song, Wagner, and Tian ’01]

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Side Channels

TECH

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Intel sells off for a second day as massive security exploit shakes the stock

- Newly discovered vulnerabilities could theoretically allow a hacker to steal information stored in the memory of chips themselves.
- Although the exploits affected leading processors in many devices, Intel is bearing most of the fallout.
- Some on Wall Street think that Intel's loss could mean gains for rivals.

Anita Balakrishnan | @MsABalakrishnan

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CNBC

- *Meltdown* (Lipp *et al.*, '18)
- *Spectre* (Kocher *et al.*, '18)

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How to measure leakage in this context?

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Given RVs X and Y, how much does Y leak about X?

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Wagner and Eckhoff ('15):

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Wagner and Eckhoff ('15): 81 metrics

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randomized/blurred version of X

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brute-force attack

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(U,X) joint distribution is complicated; "future-proof"

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Maximal Leakage

U	X	Y
[sensitive info]	[nominal process]	[revealed process]

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Markov chain: $U \longleftrightarrow X \longleftrightarrow Y$

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[operationally interpretable]

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[not evidently computable; Carathéodory?]

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[depends on P_X only through its support]

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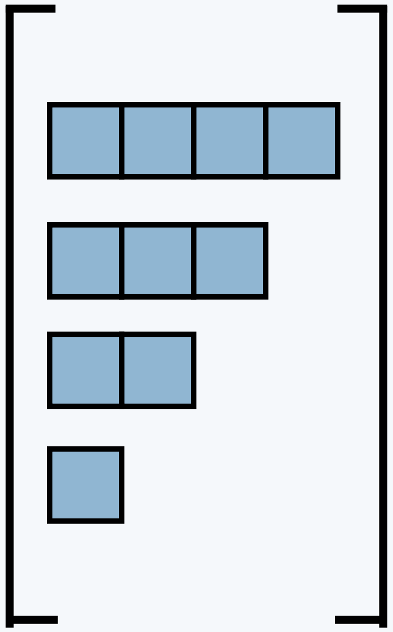

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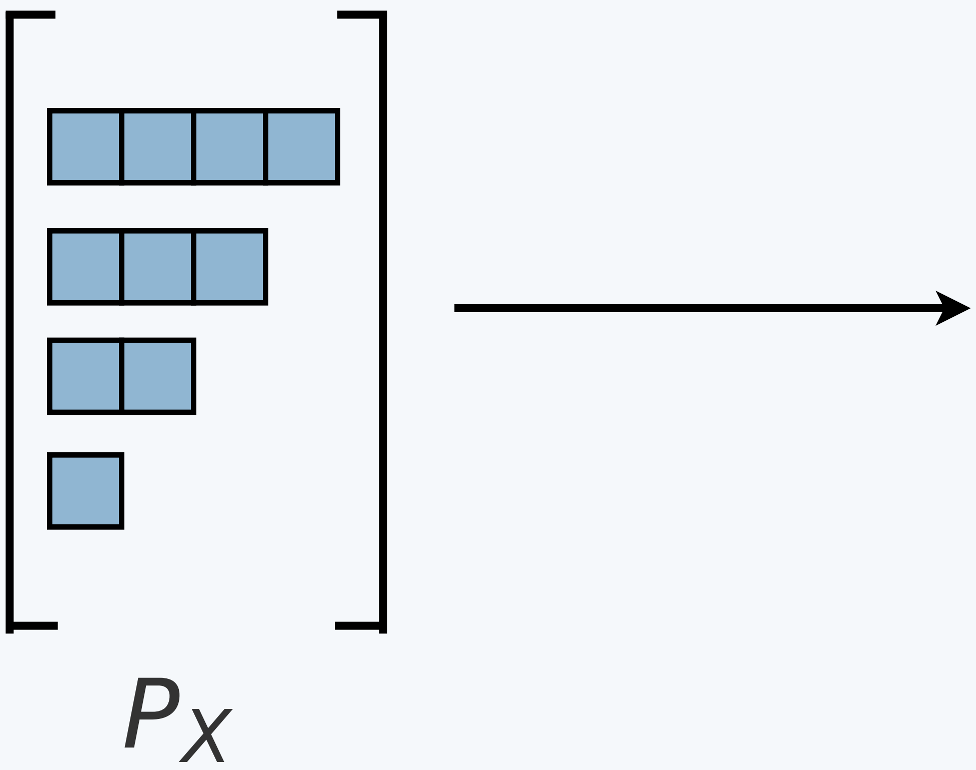

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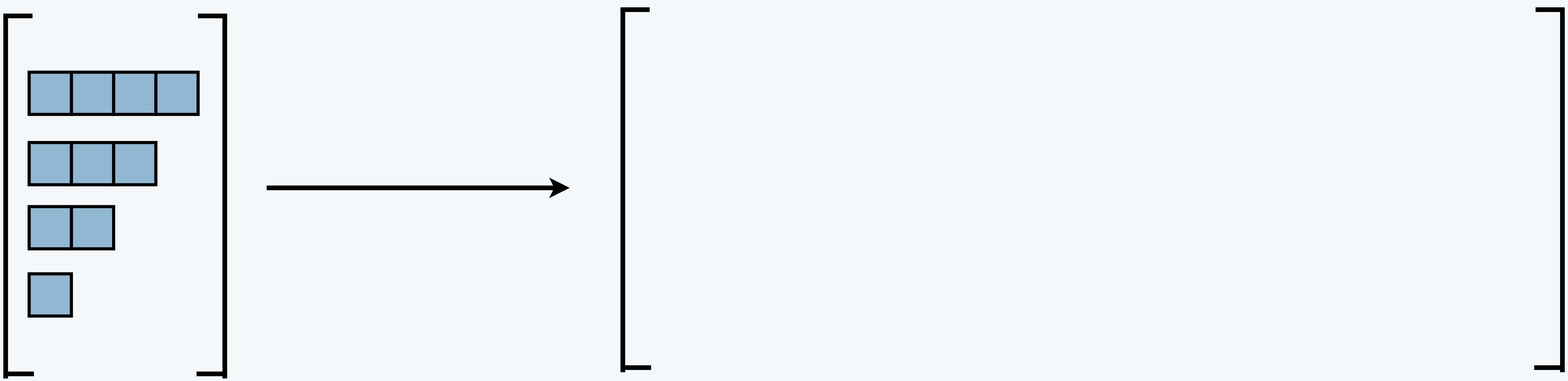

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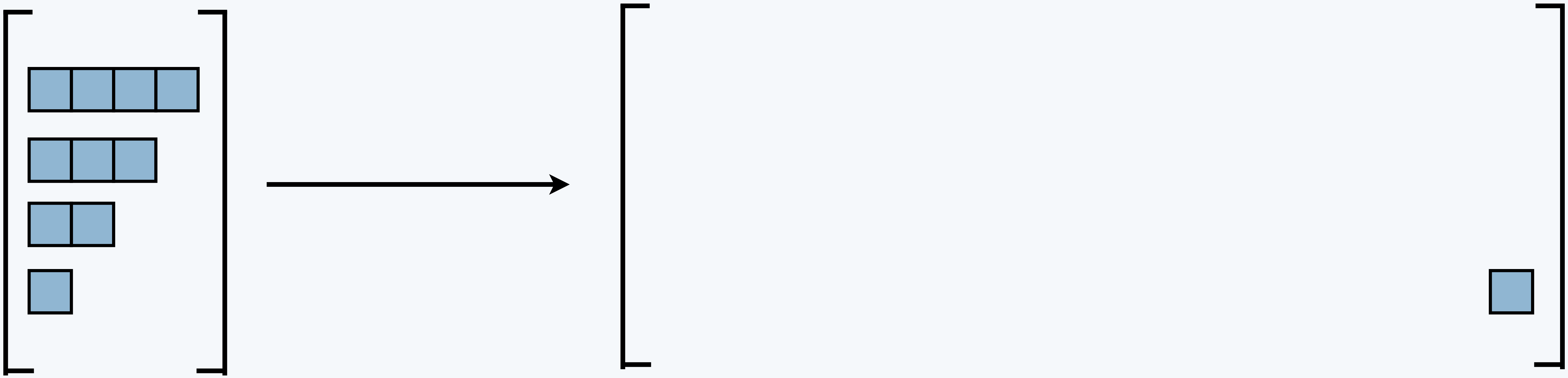

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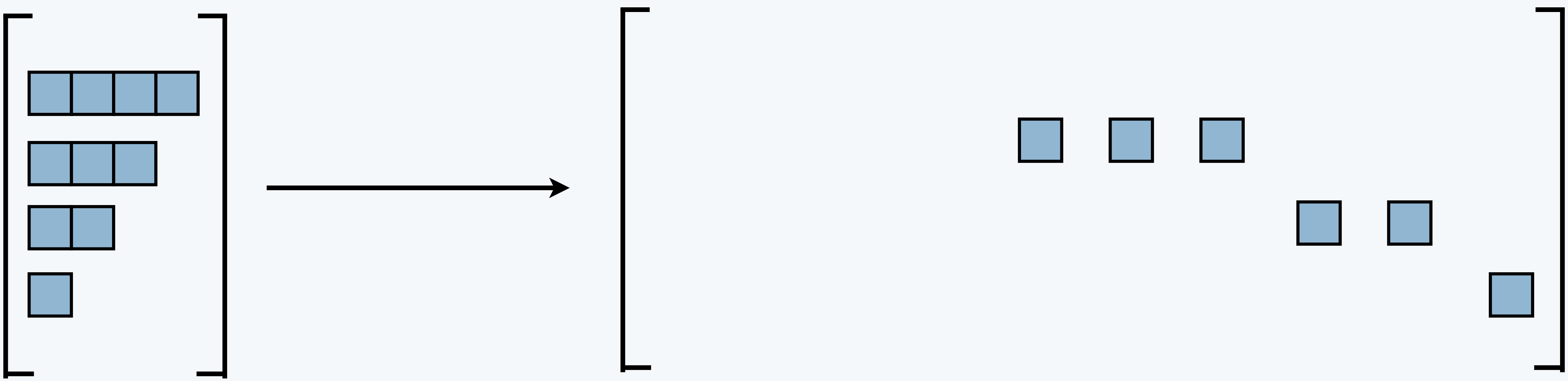

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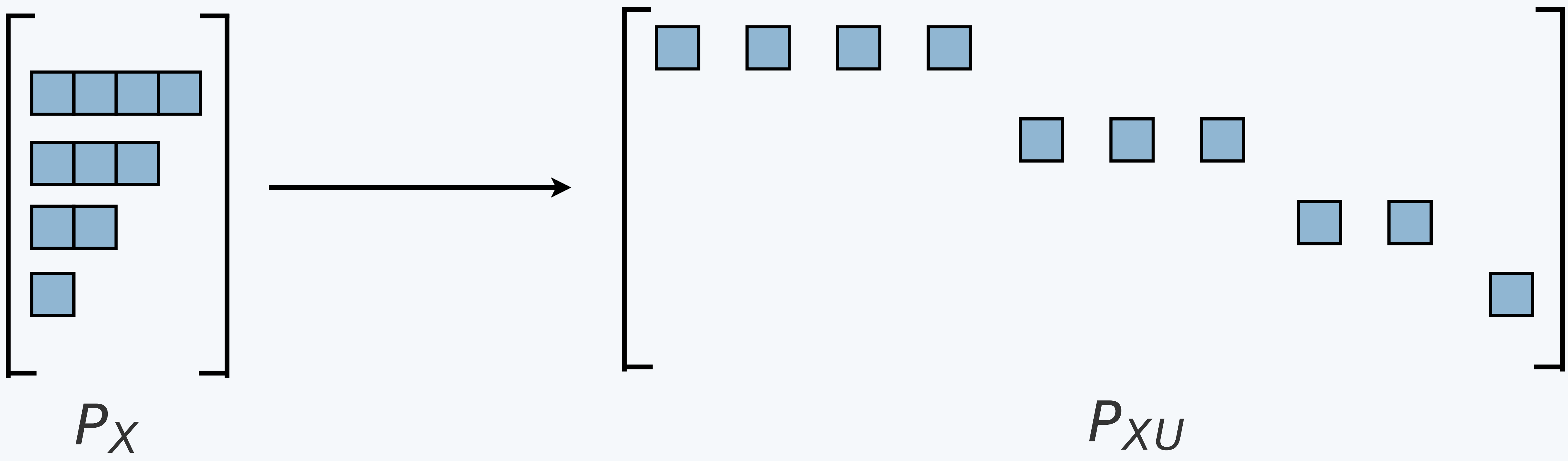

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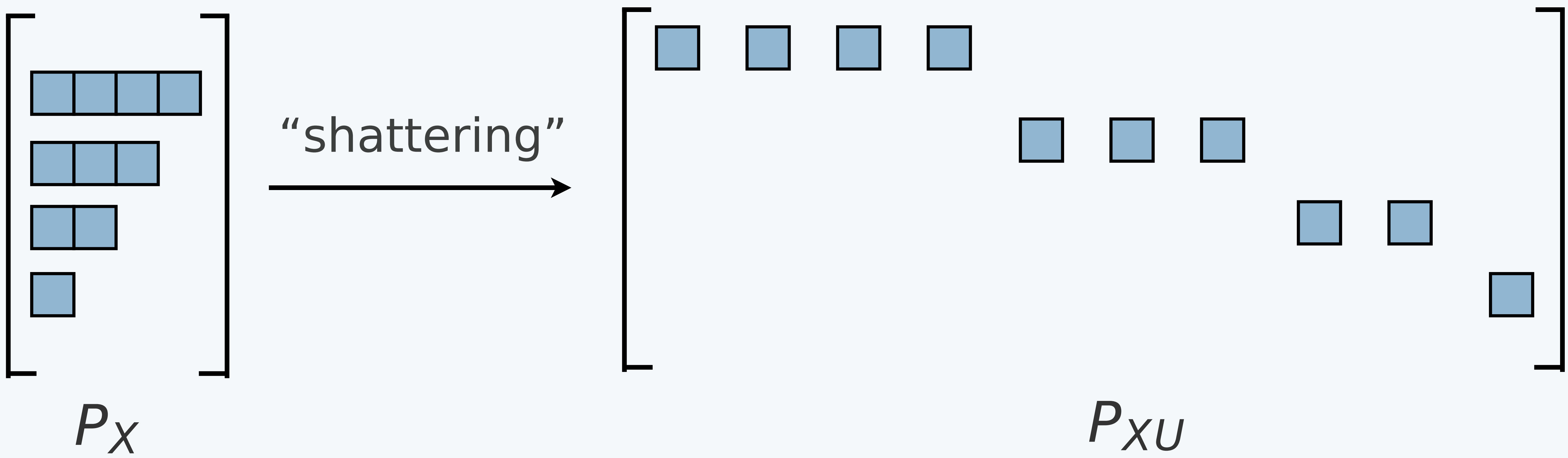

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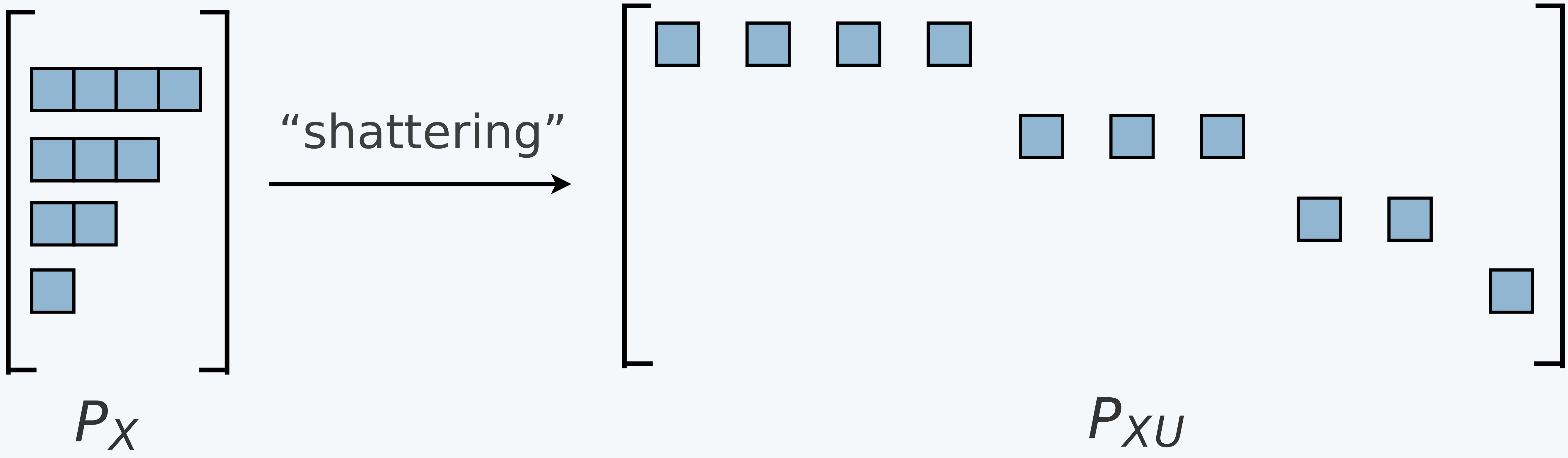

[U is uniform and s.t. X is a deterministic function of U]

Upper Bound

$$
\begin{aligned}
&\sum_{y\in\mathcal{Y}} P_Y(y) \max_{u\in\mathcal{U}} P_{U|Y}(u|y) \\
&= \sum_{y\in\mathcal{Y}} \max_{u\in\mathcal{U}} P_{UY}(u, y) \\
&= \sum_{y\in\mathcal{Y}} \max_{u\in\mathcal{U}} \sum_{x\in\mathcal{X}} P_X(x) P_{U|X}(u|x) P_{Y|X}(y|x) \\
&\leq \sum_{y\in\mathcal{Y}} \max_{u\in\mathcal{U}} \sum_{x\in\mathcal{X}} P_X(x) P_{U|X}(u|x) \max_{x'\in\mathcal{X}} P_{Y|X}(y|x') \\
&= \sum_{y\in\mathcal{Y}} \left(\max_{x'\in\mathcal{X}} P_{Y|X}(y|x') \right) \max_{u\in\mathcal{U}} \sum_{x\in\mathcal{X}} P_X(x) P_{U|X}(u|x) \\
&= \sum_{y\in\mathcal{Y}} \max_{x\in\mathcal{X}} P_{Y|X}(y|x) \max_{u\in\mathcal{U}} P_U(u).
\end{aligned}
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Theorem (Issa-Kamath-Wagner): For any joint distribution P_{XY} on finite alphabets

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- Cardinality bound

$$\mathcal{L}(X \to Y) \leq \min\{\log |\mathcal{X}|, \log |\mathcal{Y}|\}$$

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- Independence: $\mathcal{L}(X \rightarrow Y) = 0$ iff X and Y are indep.

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- Additivity: if $(X_i, Y_i)_{i=1}^n$ are independent over i

$$\mathcal{L}(X^n \to Y^n) = \sum_{i=1}^{n} \mathcal{L}(X_i \to Y_i)$$

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- Convexity: $\exp(\mathcal{L}(X \to Y))$ is convex in $P_{Y|X}$
- Maximal leakage upper bounds mutual info.

$$\mathcal{L}(X \to Y) \geq I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

Variations and Extensions

- Multiple guesses
- Approximate guesses
- General gains
- Opportunistic choice of U
- Conditional version
- Formula for general measure spaces
- Guessing X itself

Extension: Multiple Guesses

Def (Issa-Kamath-Wagner)**:** For any positive integer k,

$$\mathcal{L}_k(X \to Y) = \sup_{U \longleftrightarrow X \longleftrightarrow Y} \log \frac{\sup_{\tilde{u}_1(\cdot),\ldots,\tilde{u}_k(\cdot)} P(\cup_i \{U = \tilde{u}_i(Y)\})}{\sup_{\tilde{u}_1,\ldots,\tilde{u}_k} P(\cup_i \{U = \tilde{u}_i\})}$$

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Theorem (Issa-Kamath-Wagner)**:** If X and Y are discrete then for any positive integer k,

$$\mathcal{L}_k(X \to Y) = \mathcal{L}_1(X \to Y) = \mathcal{L}(X \to Y).$$

Conditional Form

Definition: The conditional maximal leakage from X to Y given Z is

$$\mathcal{L}(X \to Y|Z) = \sup_{U \leftrightarrow X \leftrightarrow Y|Z} \log \frac{\sup_{\tilde{u}(\cdot,\cdot)} \Pr(U = \tilde{u}(Y, Z))}{\sup_{\tilde{u}(\cdot)} \Pr(U = \tilde{u}(Z))}$$

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vs. $U \leftrightarrow X \leftrightarrow (Y,Z)$

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Theorem (Issa-Wagner):

$$\mathcal{L}(X \to Y|Z) = \max_{z} \mathcal{L}(X \to Y|Z = z)$$

Properties of Cond. Max. Leakage

Corollary: For any joint distribution P_{XYZ} on finite alphabets

- Data processing inequality: If $X \leftrightarrow Y \leftrightarrow V|Z$ then

$$\mathcal{L}(X \to V|Z) \leq \min\{\mathcal{L}(X \to Y|Z), \mathcal{L}(Y \to V|Z)\}$$

- Cond. independence: $\mathcal{L}(X \to Y|Z) = 0$ iff

$$X \leftrightarrow Z \leftrightarrow Y$$

- Mutual information:

$$\mathcal{L}(X \to Y|Z) \geq I(X; Y|Z)$$

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Guessing X

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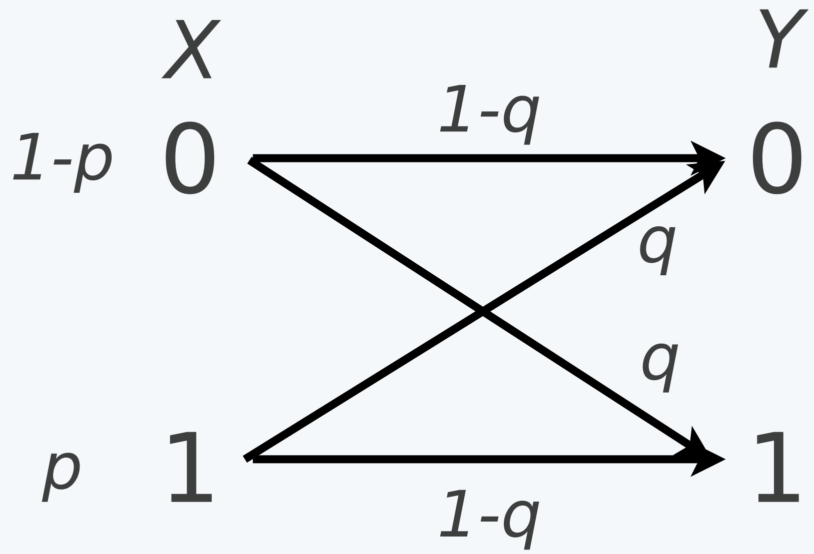
[*maximal leakage*: not in Wagner and Eckhoff ('15)]

Discrete Examples: BSC



$$\mathcal{L}(X \to Y) = \log(2(1-q))$$

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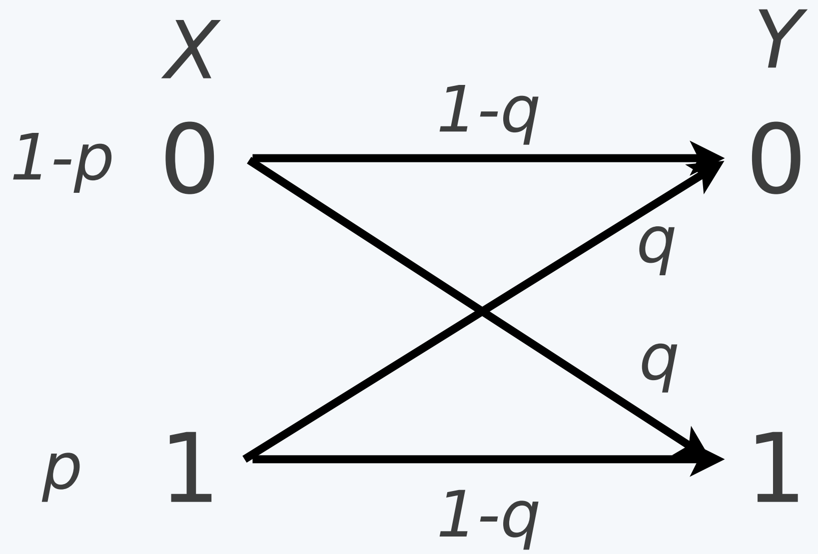

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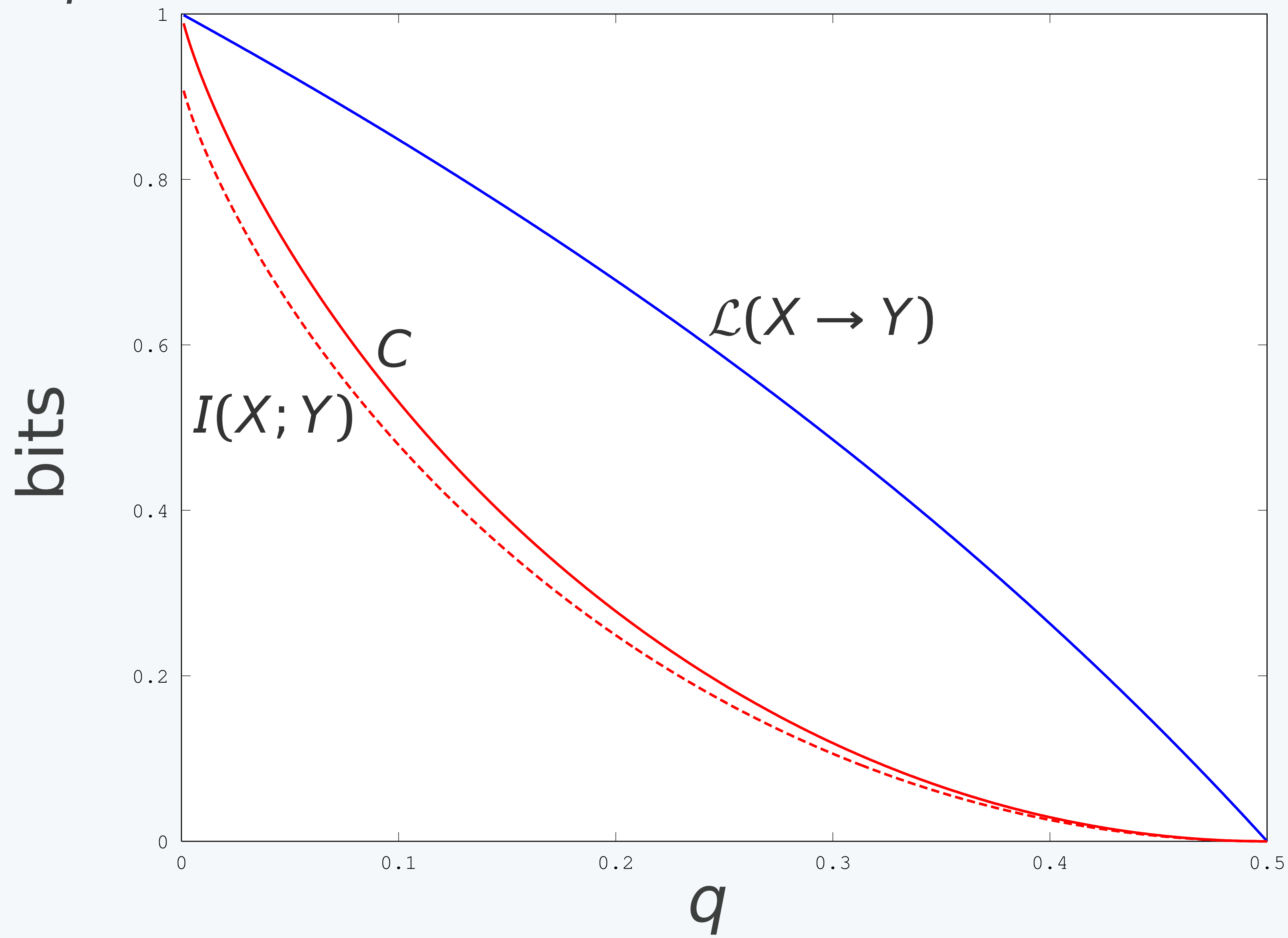

$[p = 1/3]$

Continuous Example

Theorem (Issa-Kamath-Wagner)**:** If $f_X(x)$ *and* $f_{Y|X}(y|x)$ are continuous then:

$$\mathcal{L}(X \to Y) = \log \int \sup_{x : f_X(x) > 0} f_{Y|X}(y|x)\, dy$$

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[“adding noise” (as opposed to quantizing) leaks]

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Other Metrics

- Mutual information (or equivocation)
- Expected distortion at eavesdropper
- Probability of (approximately) guessing X
- Expected number of guesses to guess X correctly
- Maximal correlation
- k-correlation
- Cryptographic advantage
- Entropic security
- (Local) differential privacy
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solution concept vs. problem formulation

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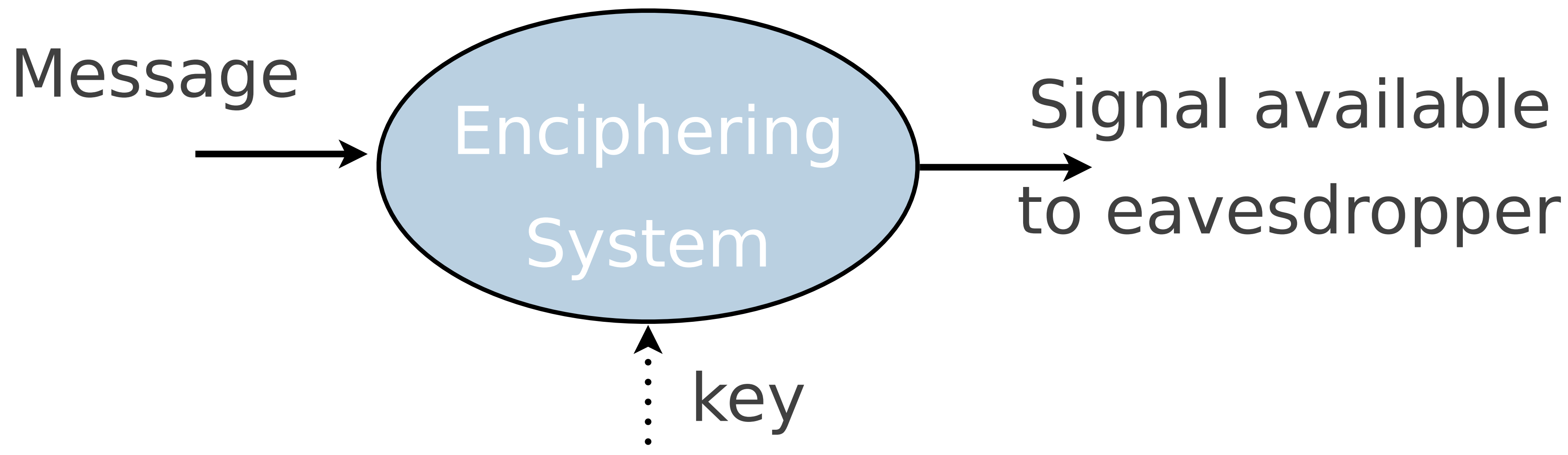
Shannon ('49):

From the point of view of the cryptanalyst, a secrecy system is almost identical with a noisy communication system. The message (transmitted signal) is operated on by a statistical element, the enciphering system, with its statistically chosen key. The result of this operation is the cryptogram (analogous to the perturbed signal) which is available for analysis. The chief differences in the two cases are: first, that the operation of the enciphering transformation is generally of a more complex nature than the perturbing noise in a channel; and, second, the key for a secrecy system is usually chosen from a finite set of possibilities while the noise in a channel is more often continually introduced, in effect chosen from an infinite set.

With these considerations in mind it is natural to use the equivocation as a theoretical secrecy index. It may be noted that there are two significant equivocations, that of the key and that of the message. These will be

Shannon ('49)

Secrecy System



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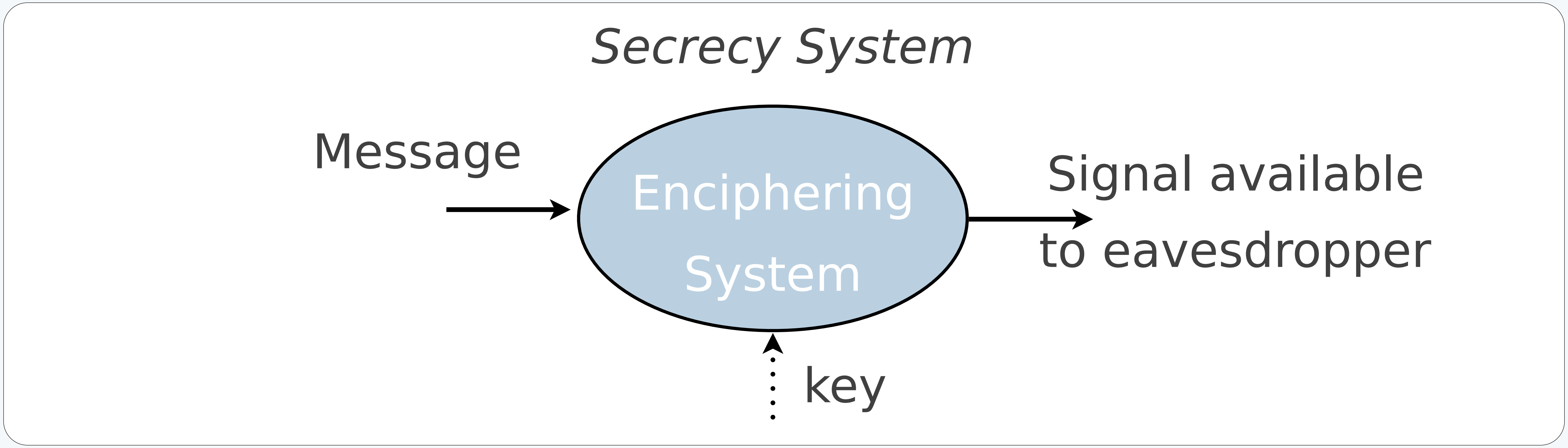
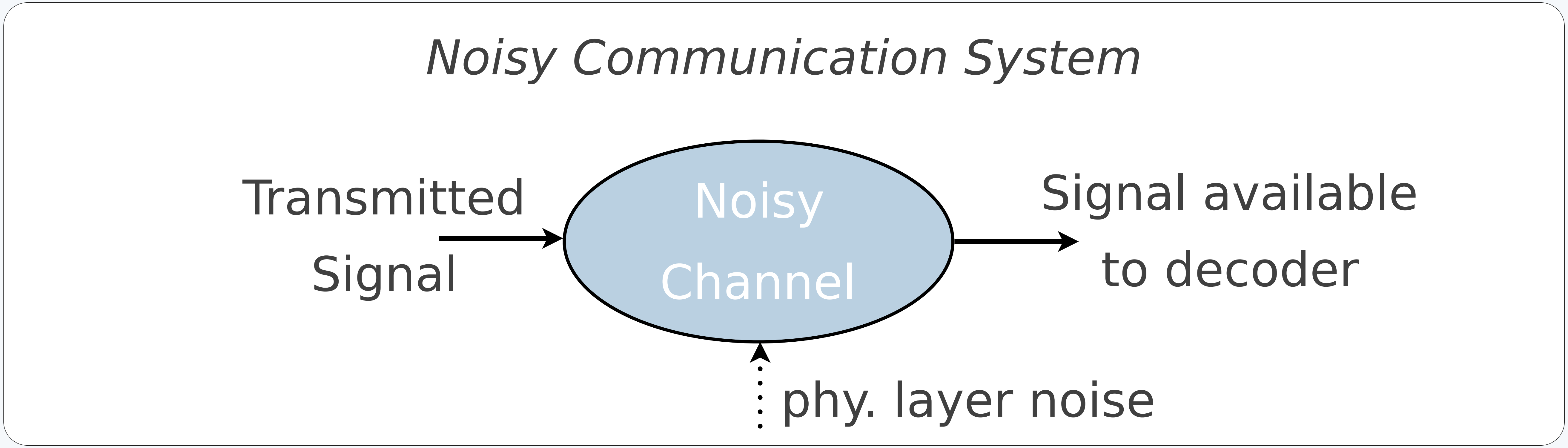

Secrecy System
Message
Enciphering
System
Signal available
to eavesdropper
key


Noisy Communication System
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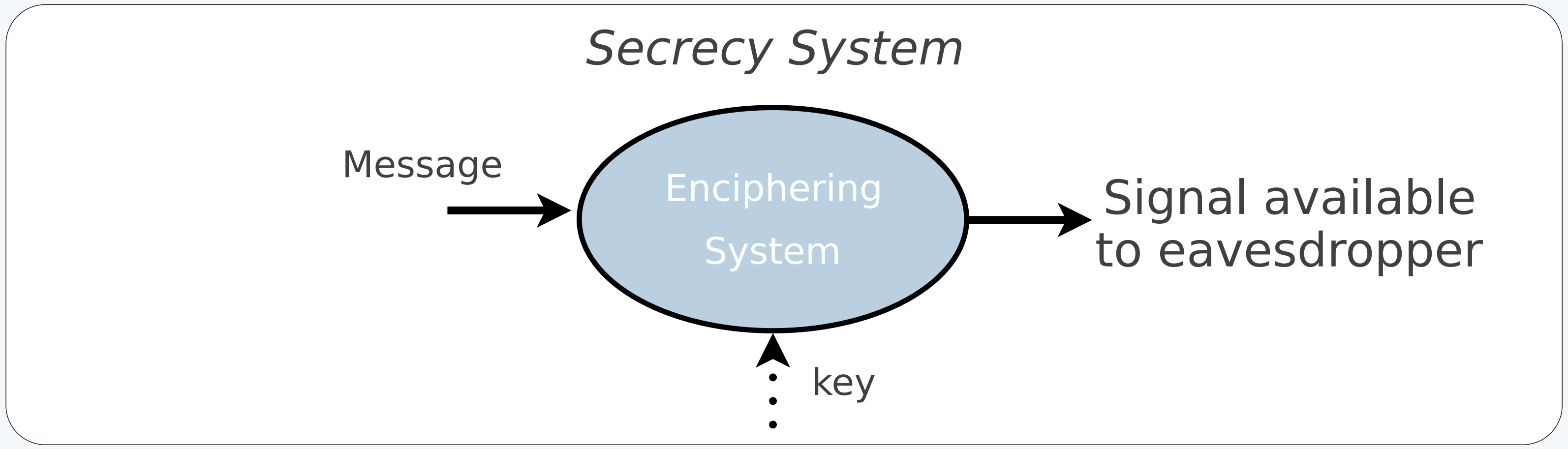

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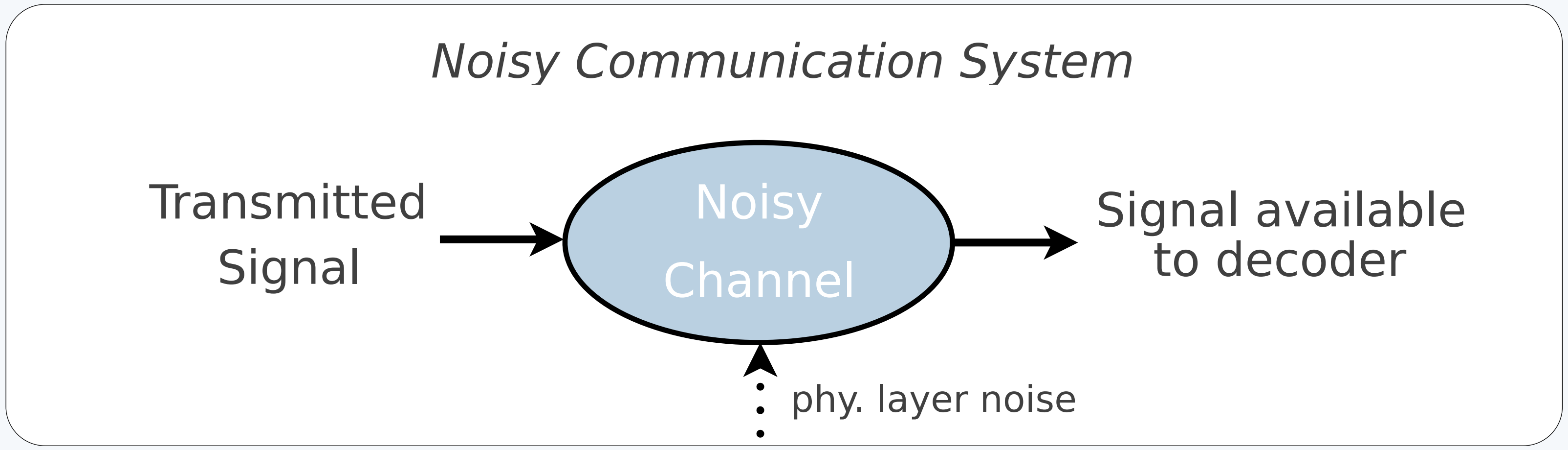

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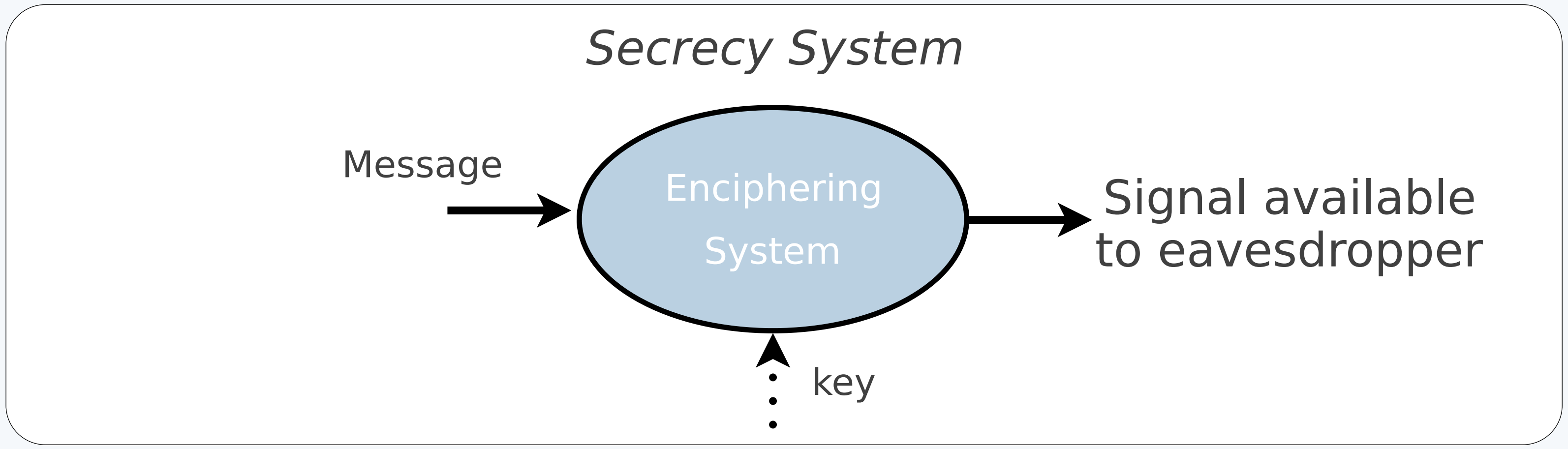

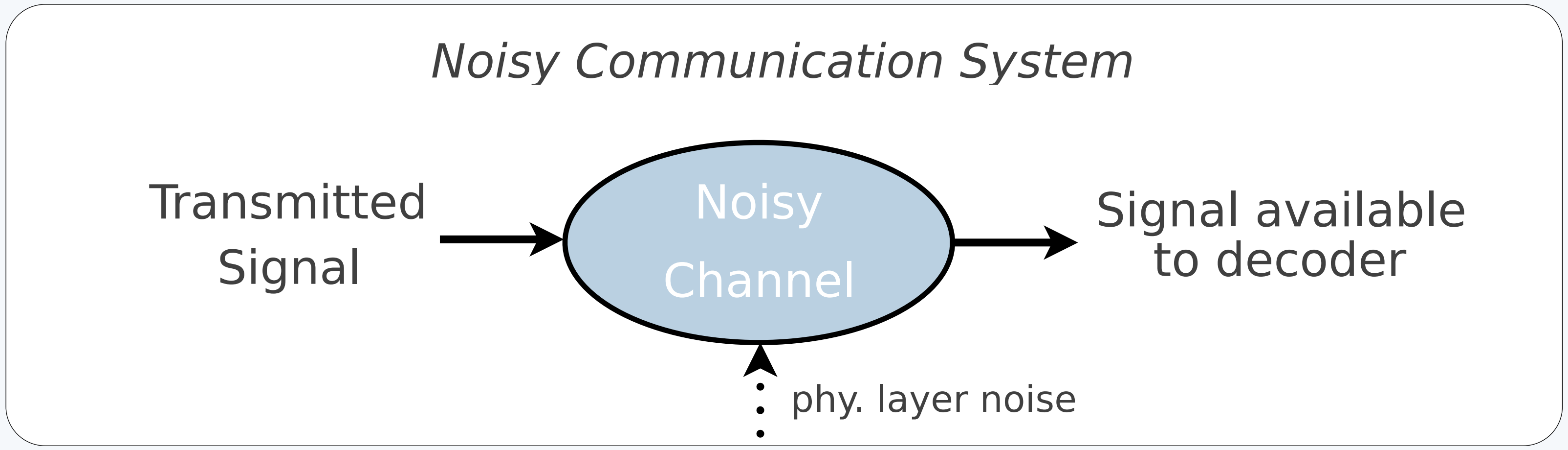

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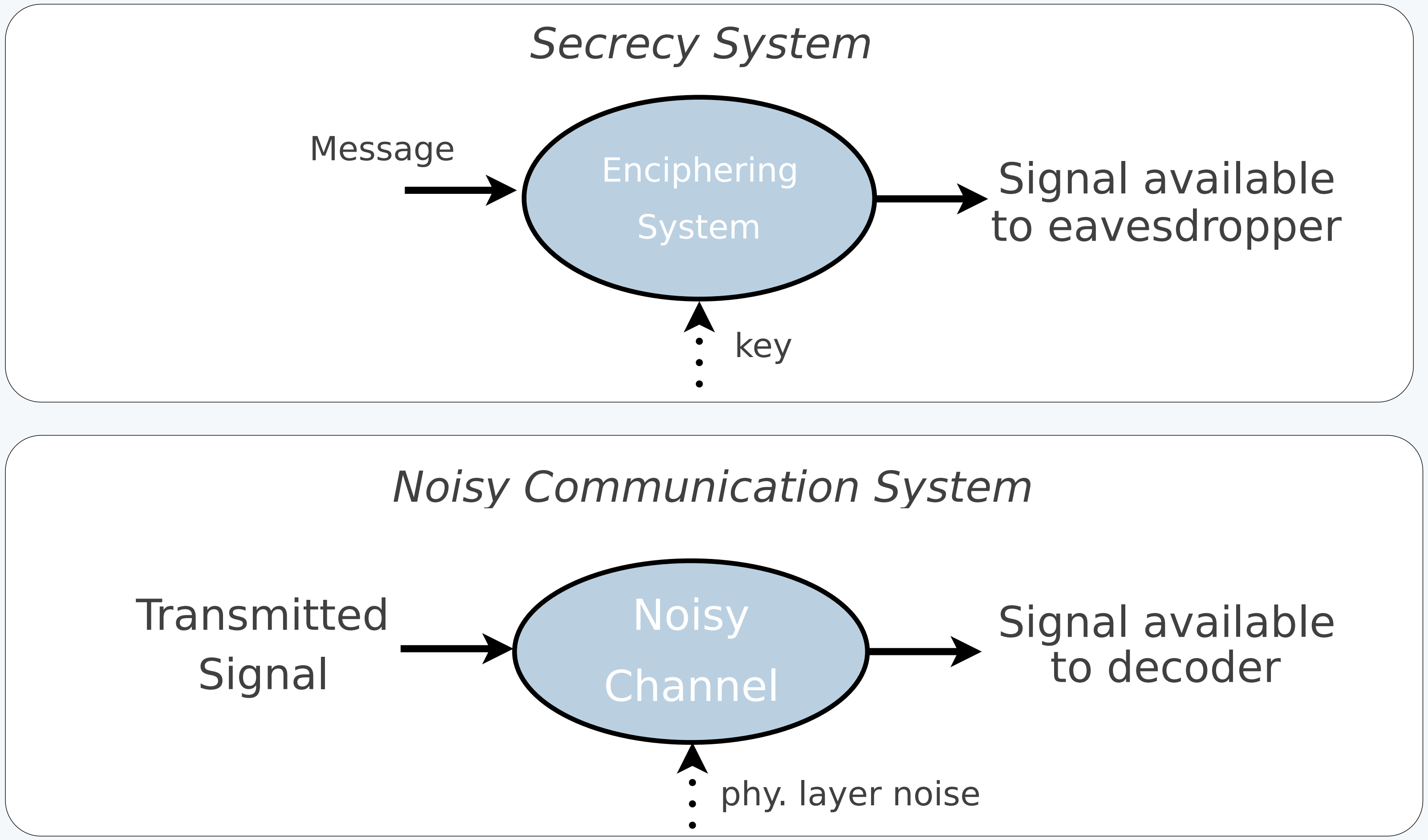

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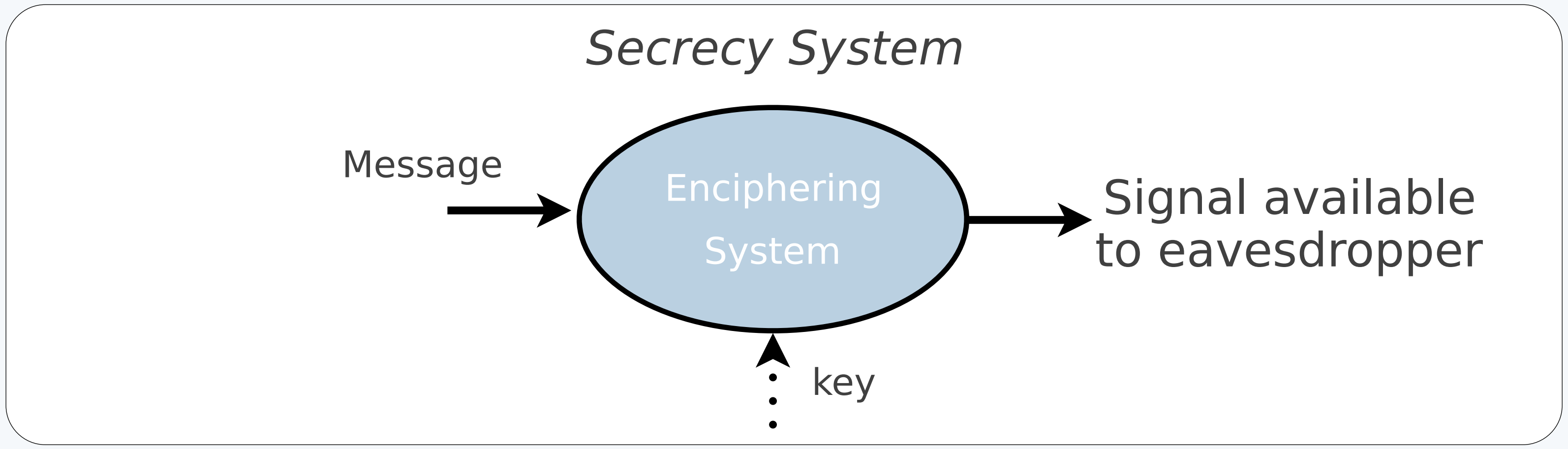

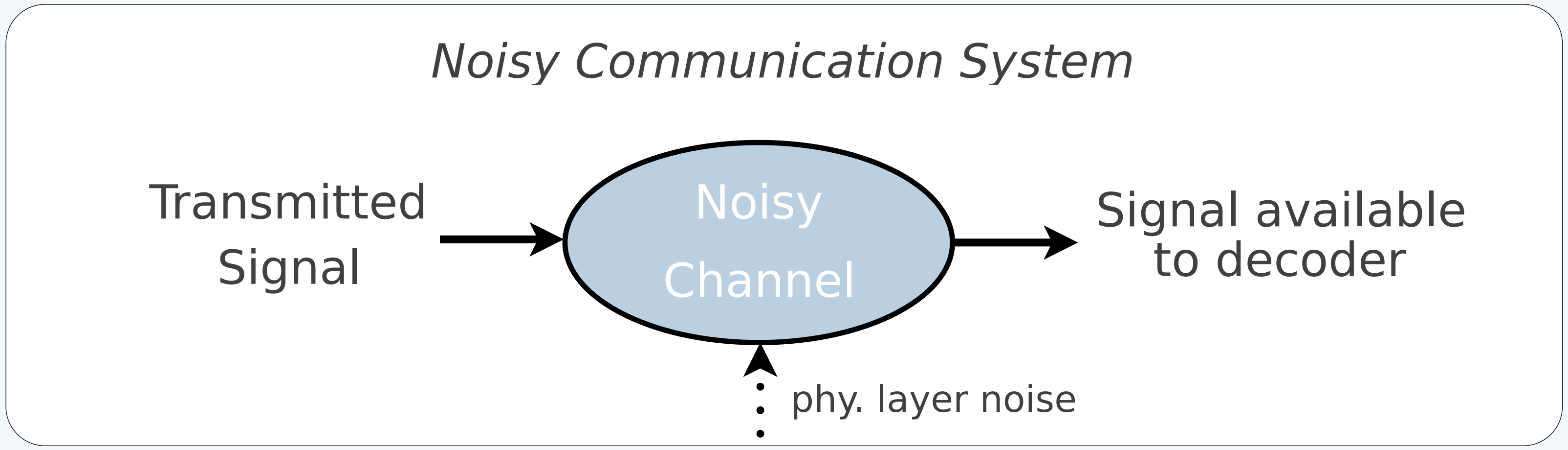

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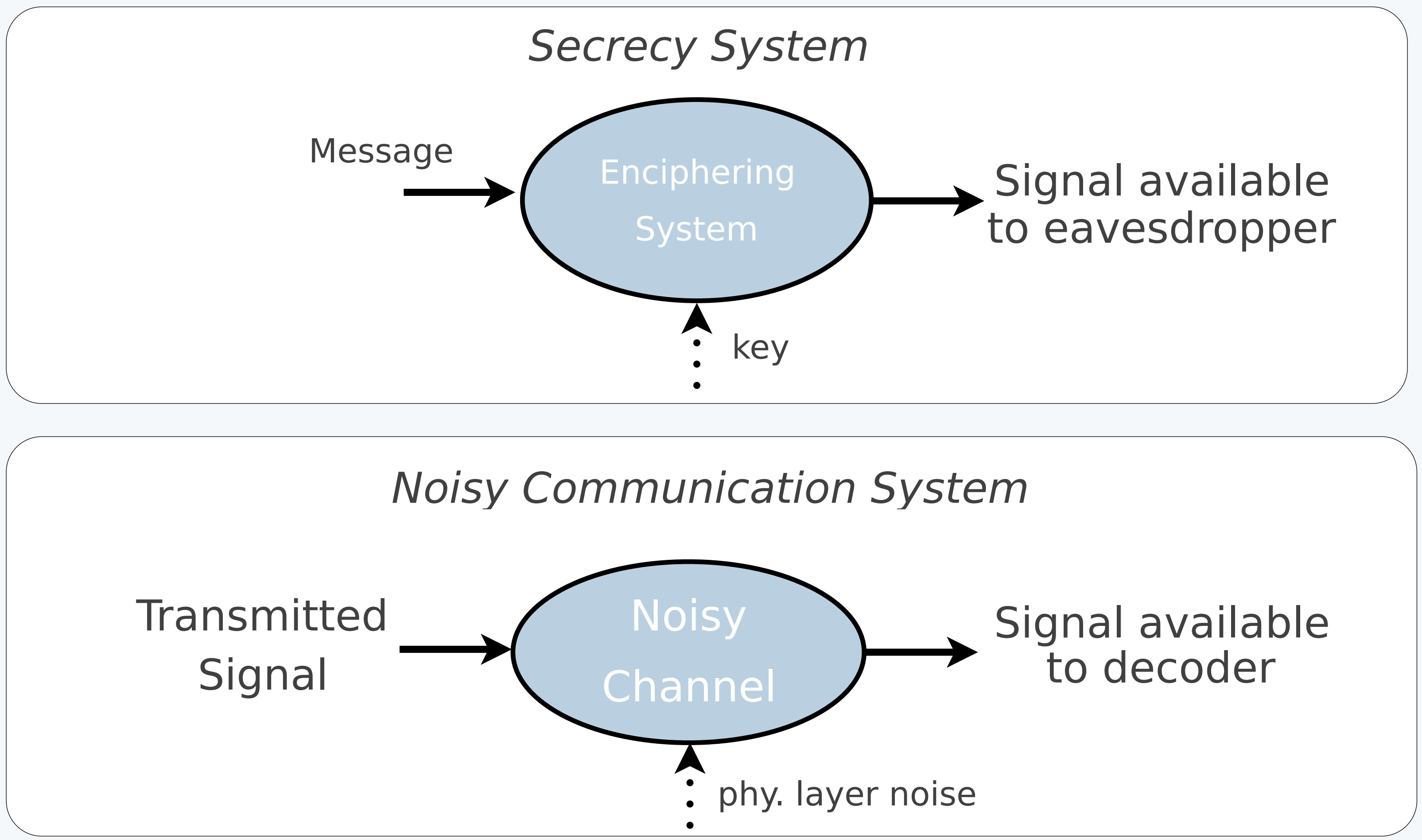

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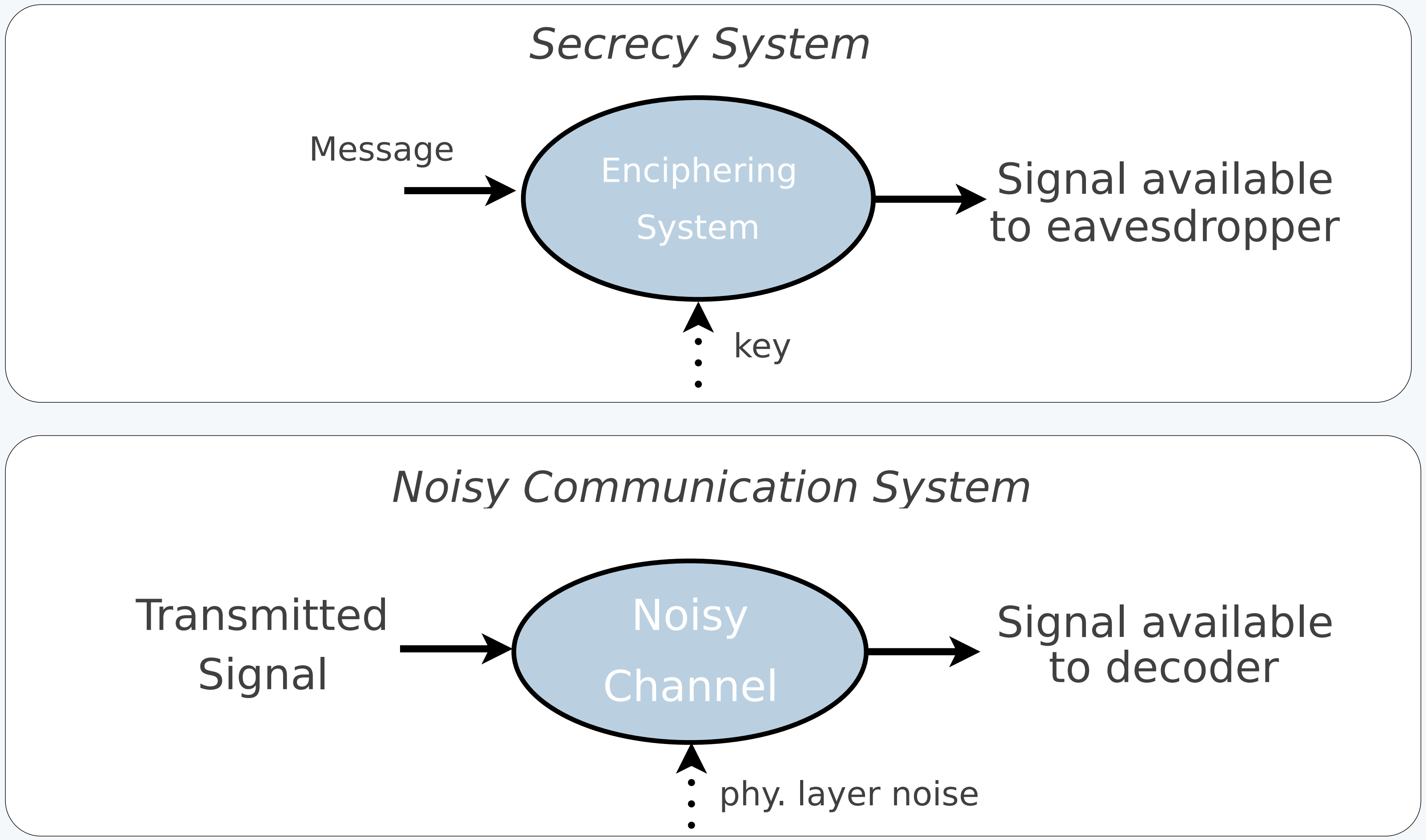

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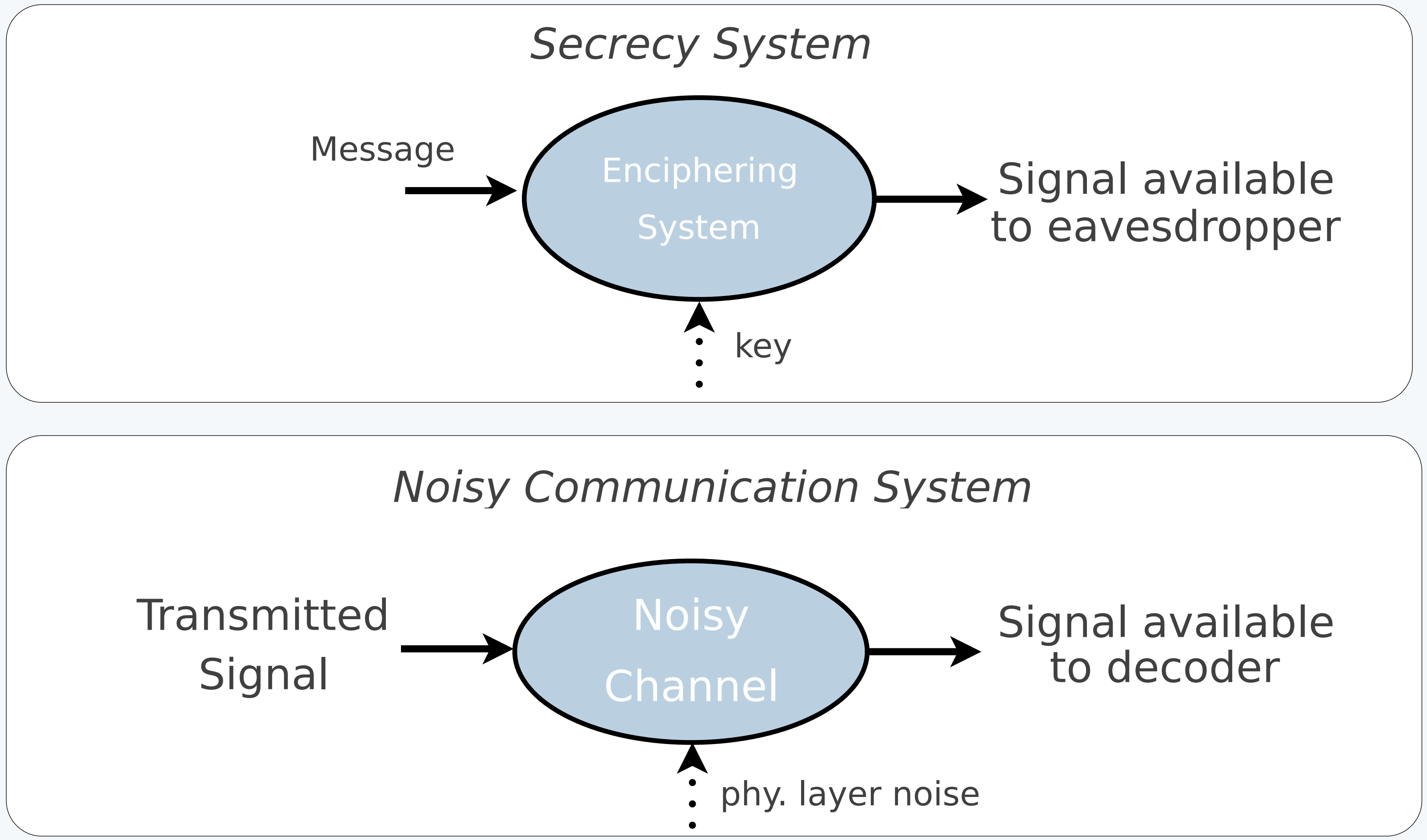

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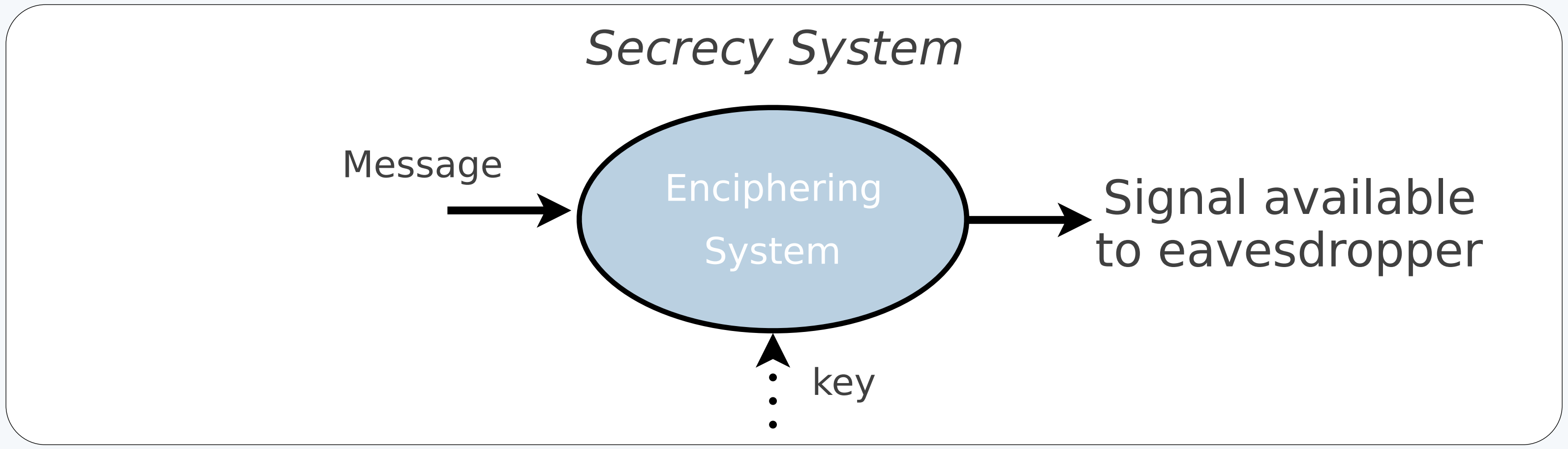

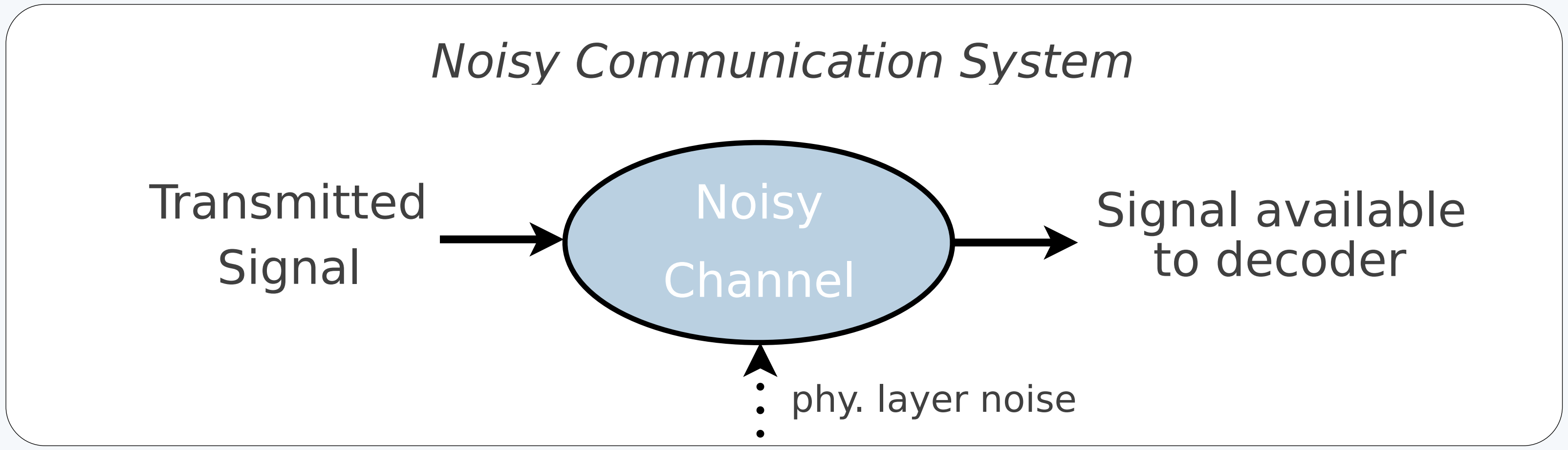

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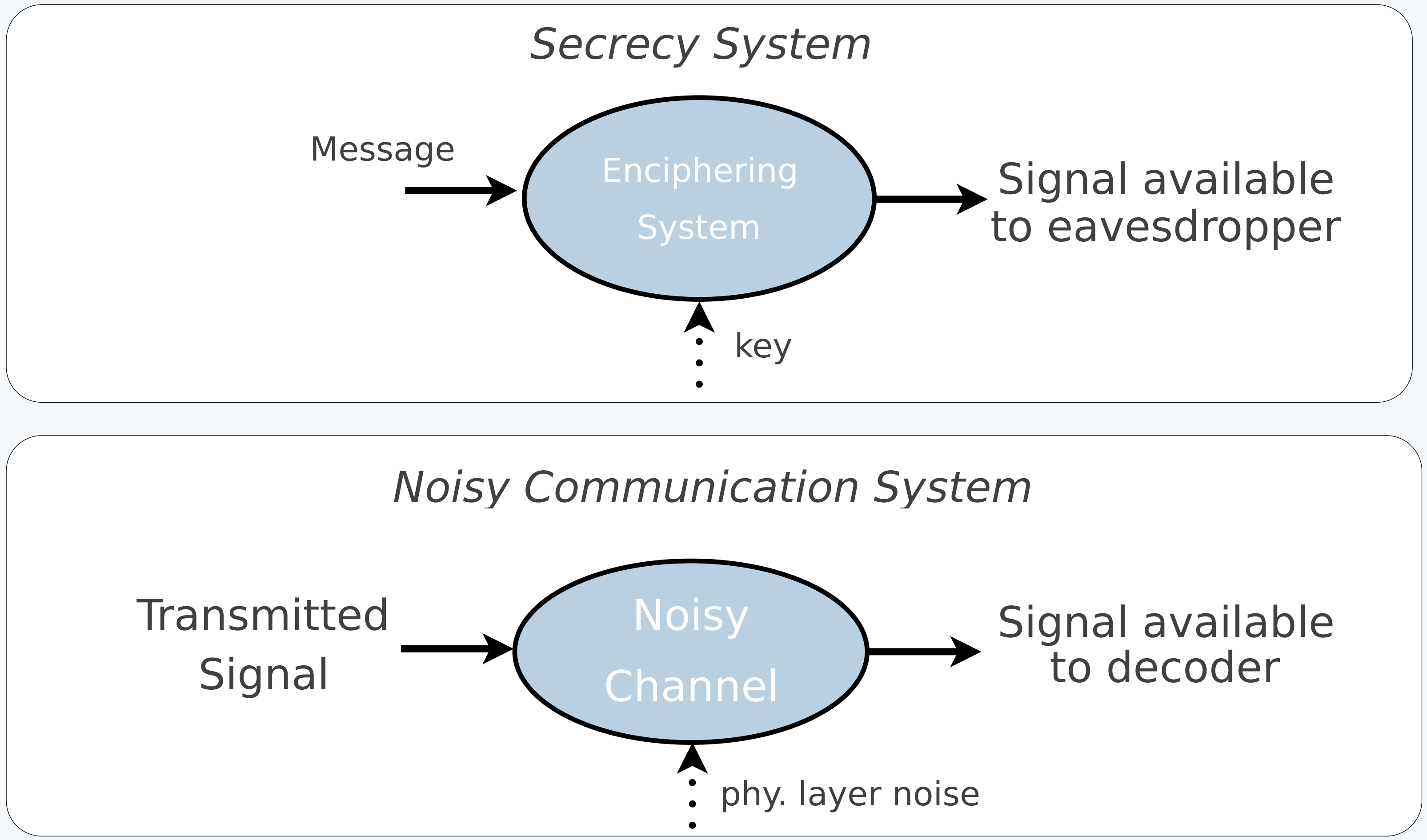

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C is the maximum amortized rate of **reliable** information transfer over a channel.

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Leakage vs. Capacity

If X has full support:

$$\mathcal{L}(X \to Y) = \sup_{U \leftrightarrow X \leftrightarrow Y} \log \frac{\sup_{\tilde{u}(\cdot)} \Pr(U = \tilde{u}(Y))}{\sup_{\tilde{u}} \Pr(U = \tilde{u})}$$

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Theorem (Issa-Wagner):

$$C = \lim_{\epsilon\to 0} \lim_{n\to\infty} \sup_{P_{X^n}} \sup_{\substack{U\leftrightarrow X^n \leftrightarrow Y^n \to \tilde{U}: \\ P(U=\tilde{U})\geq 1-\epsilon}} \frac{1}{n} \log \frac{\Pr(U = \tilde{U})}{\sup_{\tilde{u}} \Pr(U = \tilde{u})}$$

(Local) Differential Privacy

$$LDP(X \to Y) := \sup_{x,x',y} \log \frac{P_{Y|X}(y|x)}{P_{Y|X}(y|x')}$$

[Warner '65;
Evfimievski *et al.* '03]

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Optimal Mechanisms

- Given $p(x)$ and $c(x,y)$, solve

$$\min_{p(y|x)} \sum_y \max_x p(y|x)$$

$$\text{subject to} \quad \sum_{x,y} p(x)p(y|x)c(x,y) \leq C$$

$$\sum_y p(y|x) = 1 \;\; \forall\, x$$

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“exp-leakage”

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Formulation as an LP

$$
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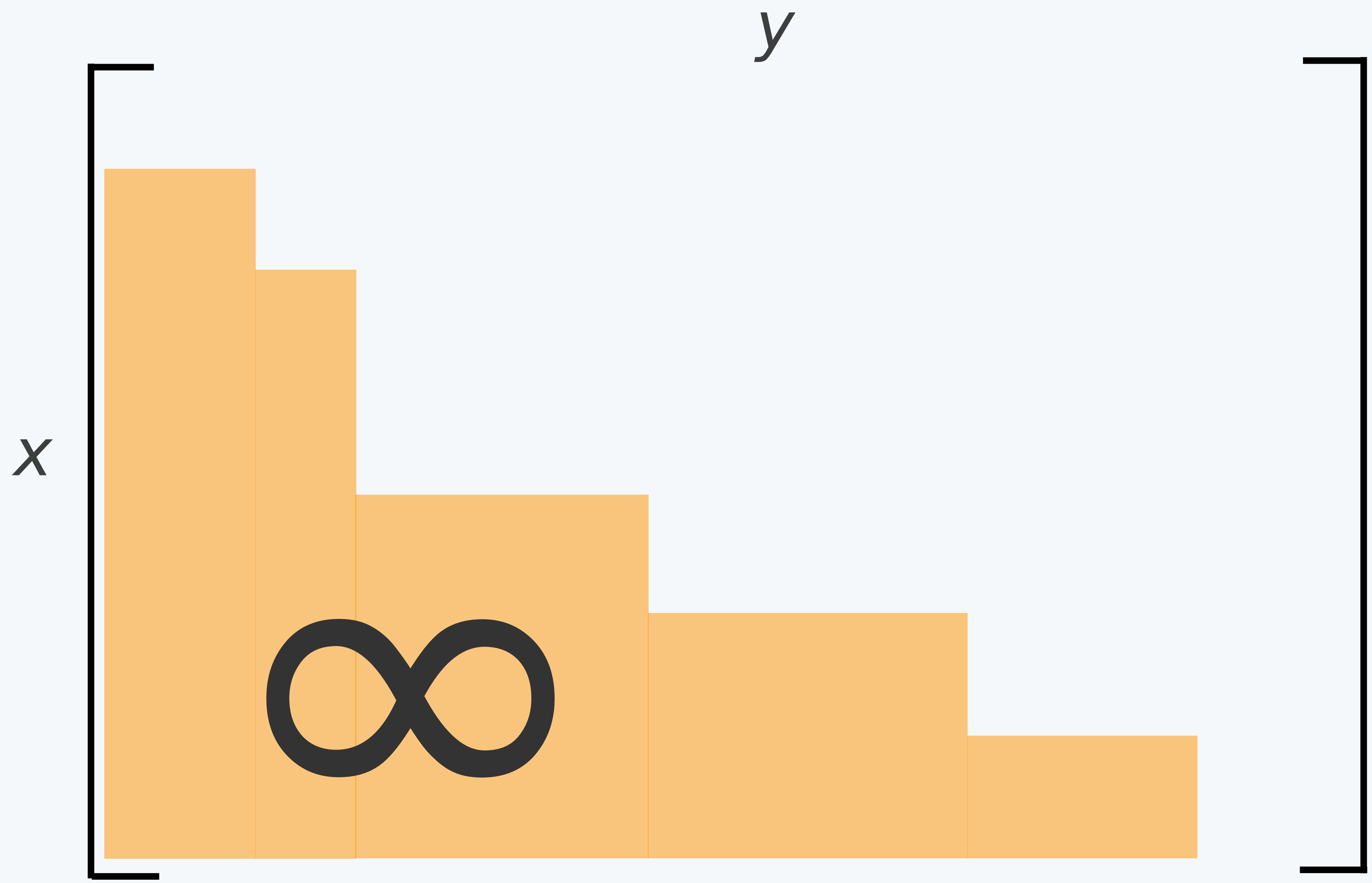
A Structural Assumption

$c(x,y)$:



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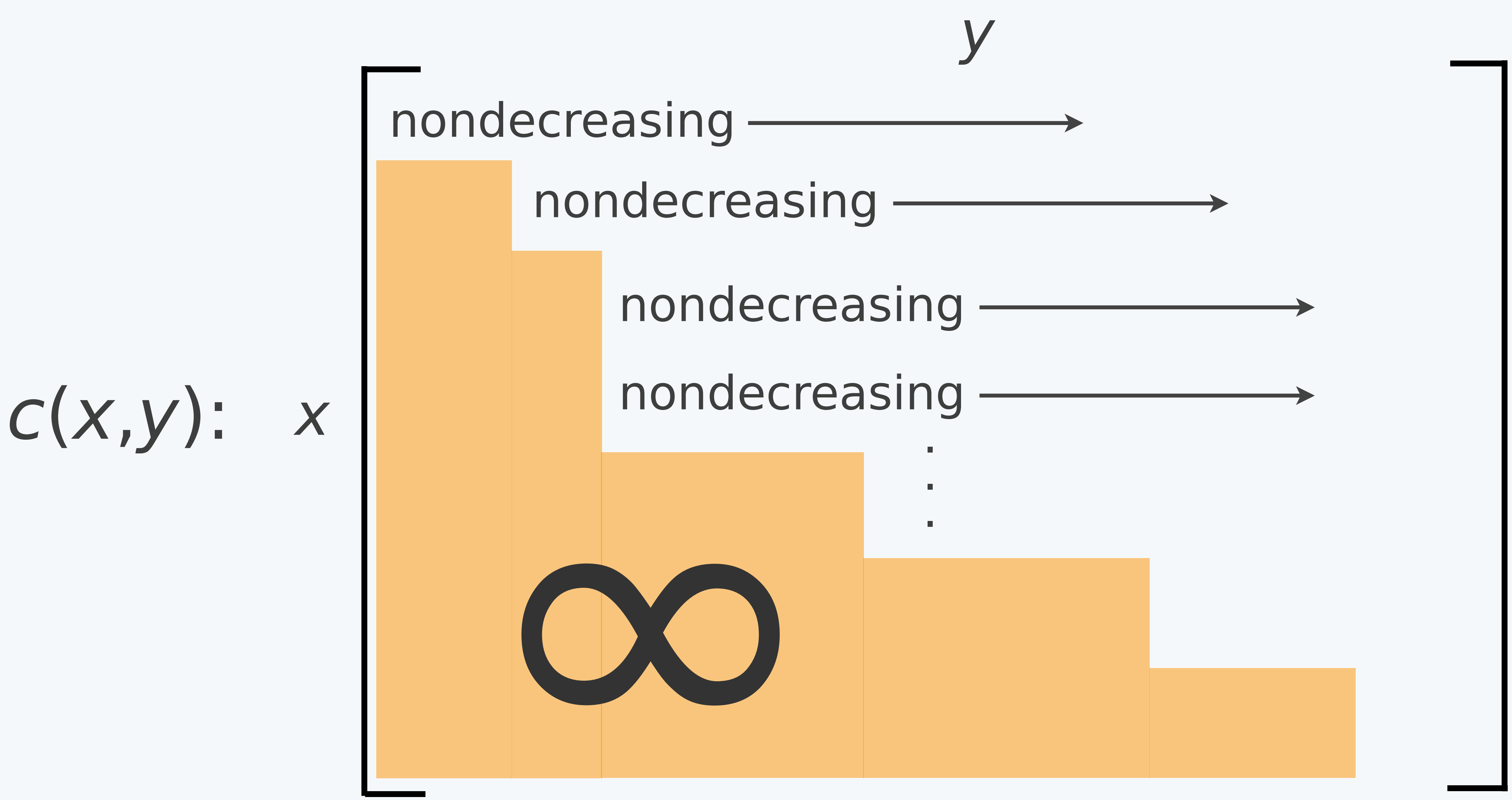

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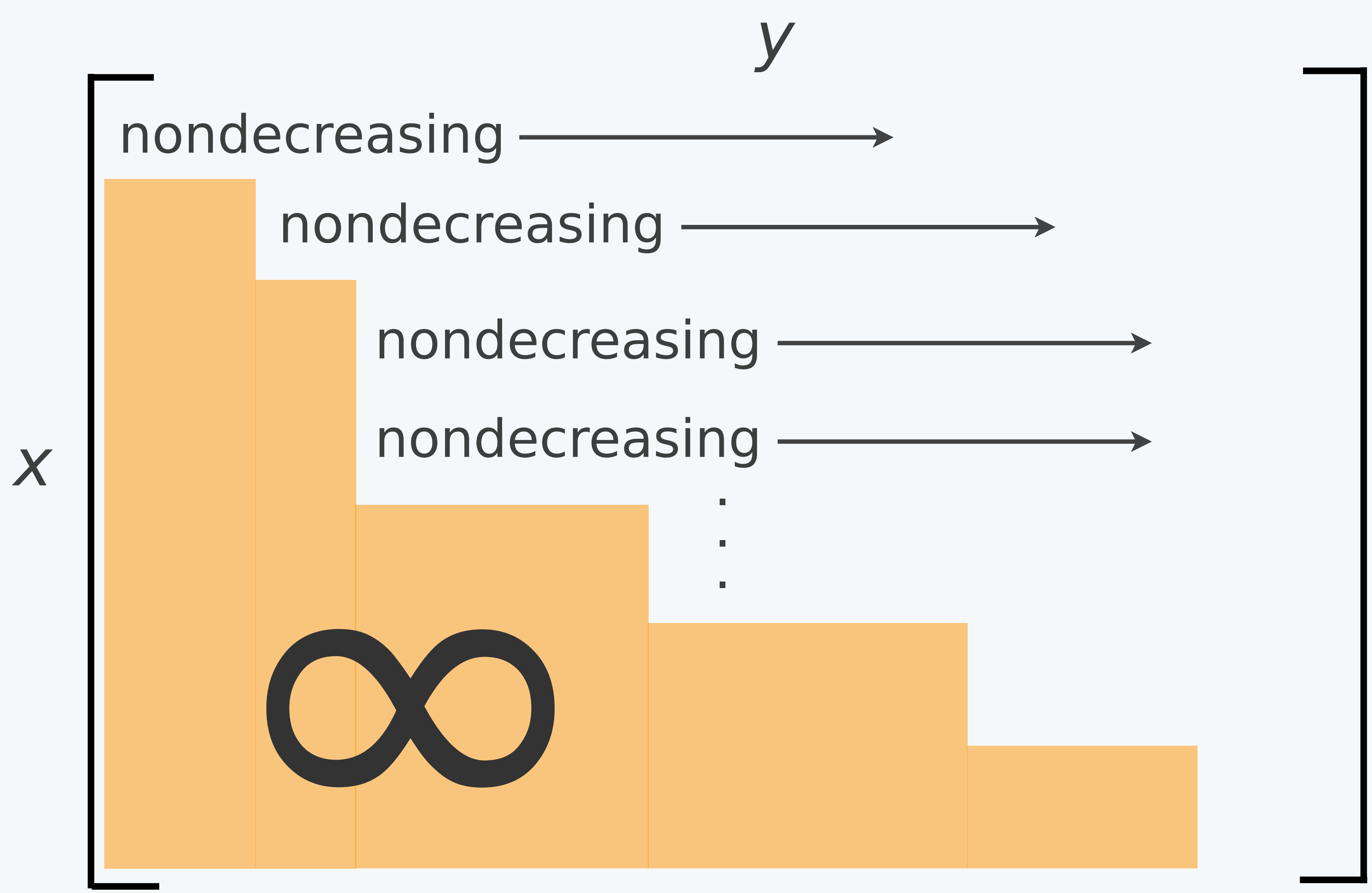
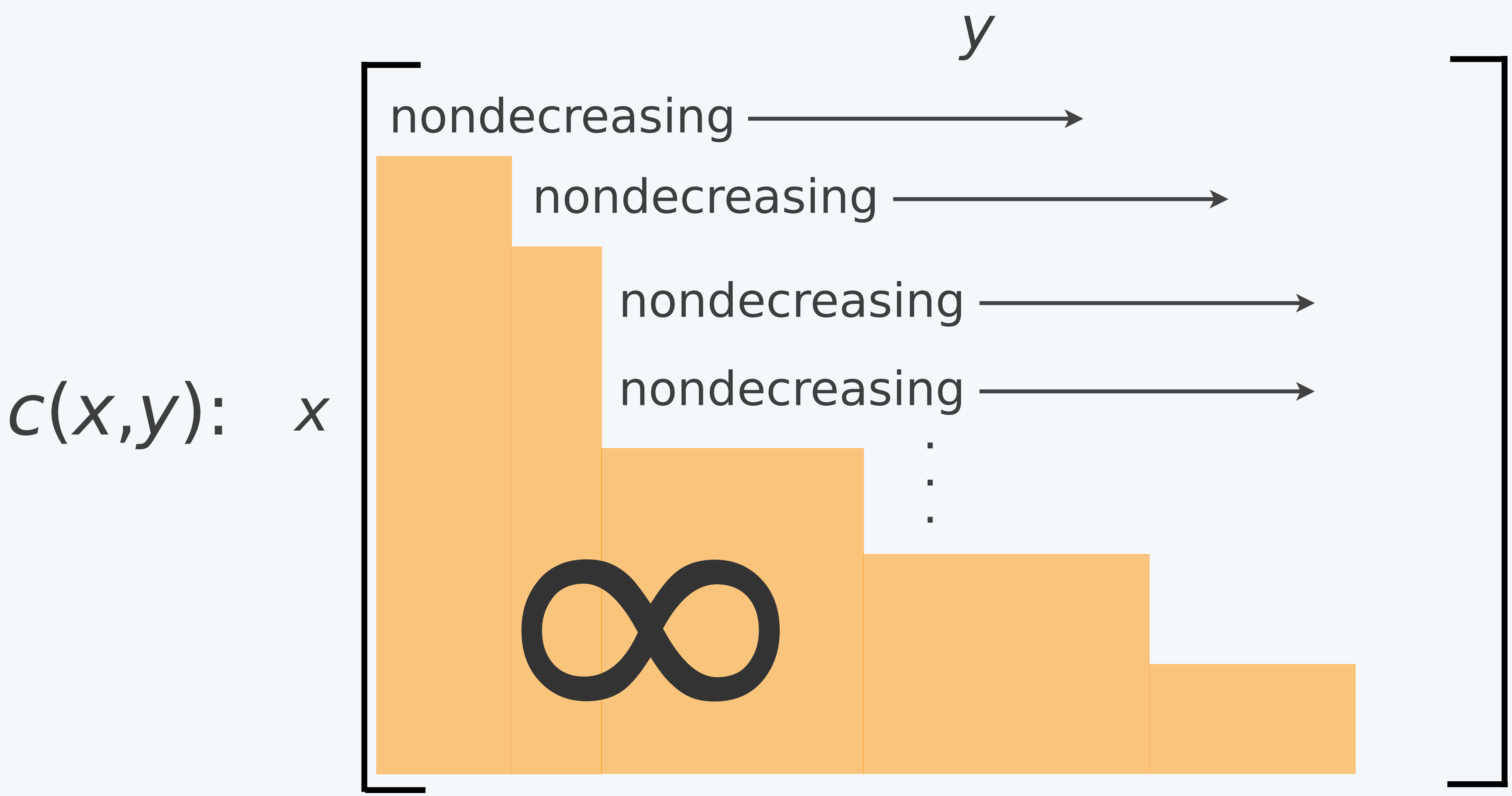


- Examples:

A Structural Assumption



- Examples:
  - Execution time [RSA], power consumption

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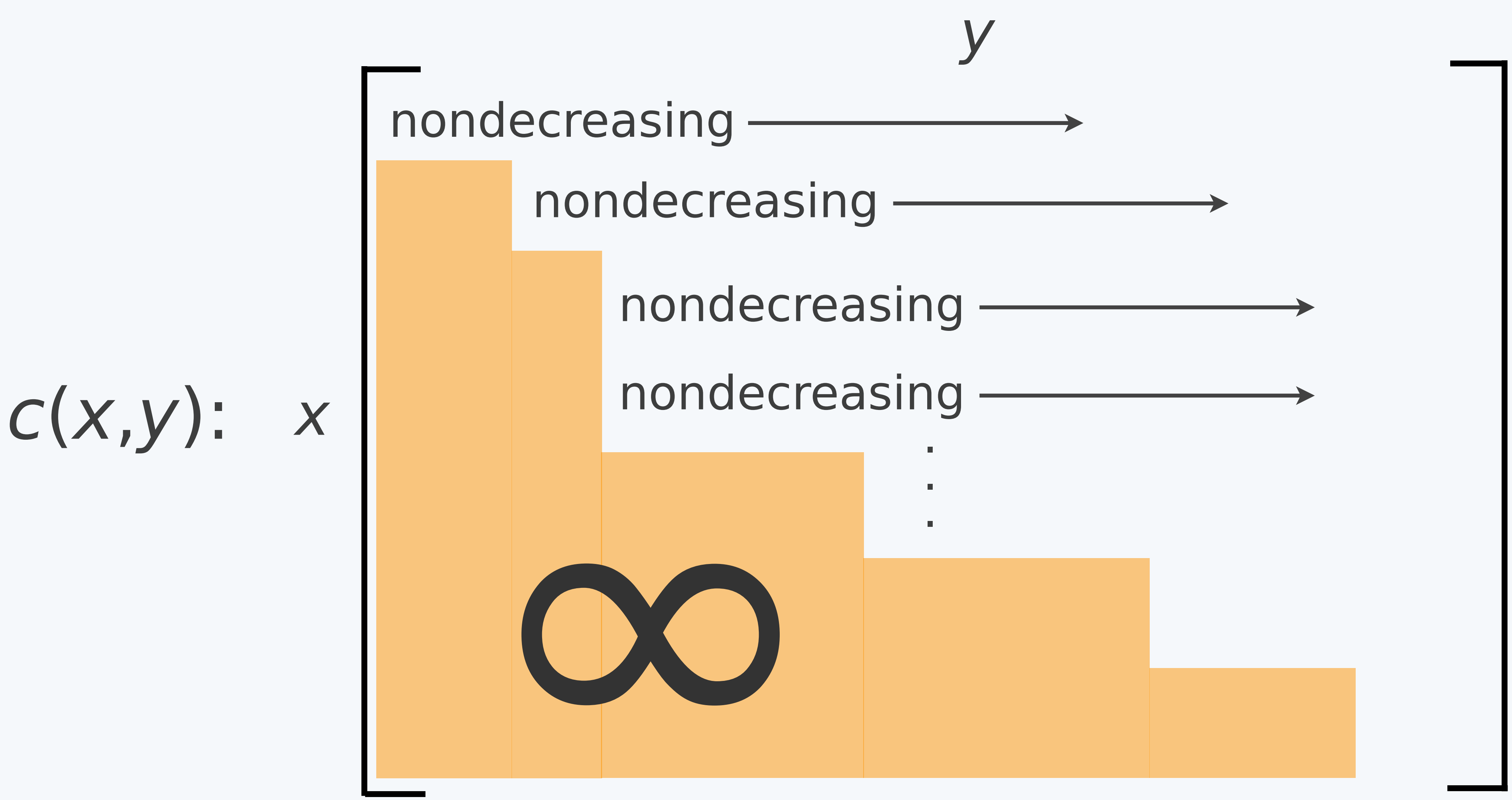

- Examples:
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Deterministic Mechanisms Are Optimal

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Theorem (Wu, Wagner, Suh):

If $c(\cdot,\cdot)$ is staircase increasing, then for any α and P_X,

$$\sum_y \max_x P_{Y|X}(y|x) + \alpha \cdot \sum_x \sum_y P_X(x) P_{Y|X}(y|x) c(x,y)$$

is minimized by a deterministic (0-1) $P_{Y|X}$.

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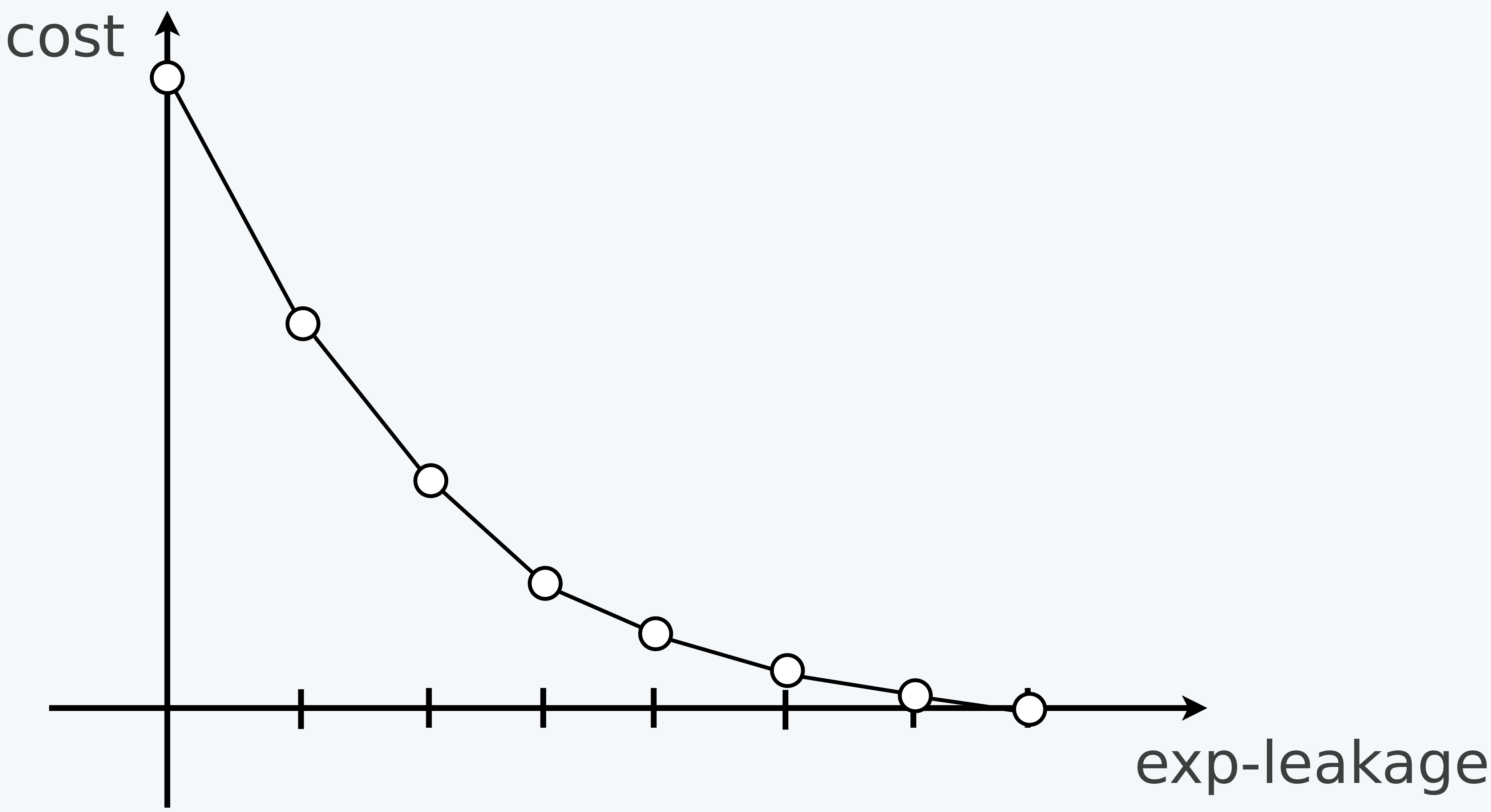
Fails for the cost matrix: $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Corollary

Corollary (Wu, Wagner, Suh):

The optimal cost/exp-leakage curve is piecewise linear with kink points only at integer exp-leakage values.



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- Immune to averaging attacks

cf. Other Metrics

$$c(x,y) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ \infty & 1 & 2 & 3 \\ \infty & \infty & 1 & 2 \\ \infty & \infty & \infty & 1 \end{bmatrix}$$

$p(x)$: uniform

minimize $\{E[c(X,Y])] : \text{leakage} \leq 1\}$

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Local Diff. Privacy:

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- Learning ML from trace data (Issa and Wagner, ’18)

Three Takeaways

Def (Issa-Kamath-Wagner)**:** Given P_{XY}, the *maximal leakage* from X to Y is

$$\mathcal{L}(X \to Y) = \sup_{U : U \leftrightarrow X \leftrightarrow Y} \log \frac{\sup_{\tilde{u}(\cdot)} \Pr(U = \tilde{u}(Y))}{\sup_{\tilde{u}} \Pr(U = \tilde{u})}$$

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Maximal leakage ...

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3. ... favors deterministic mechanisms (quantization) over "adding noise" in many contexts.

Extra Slides

A Different Question

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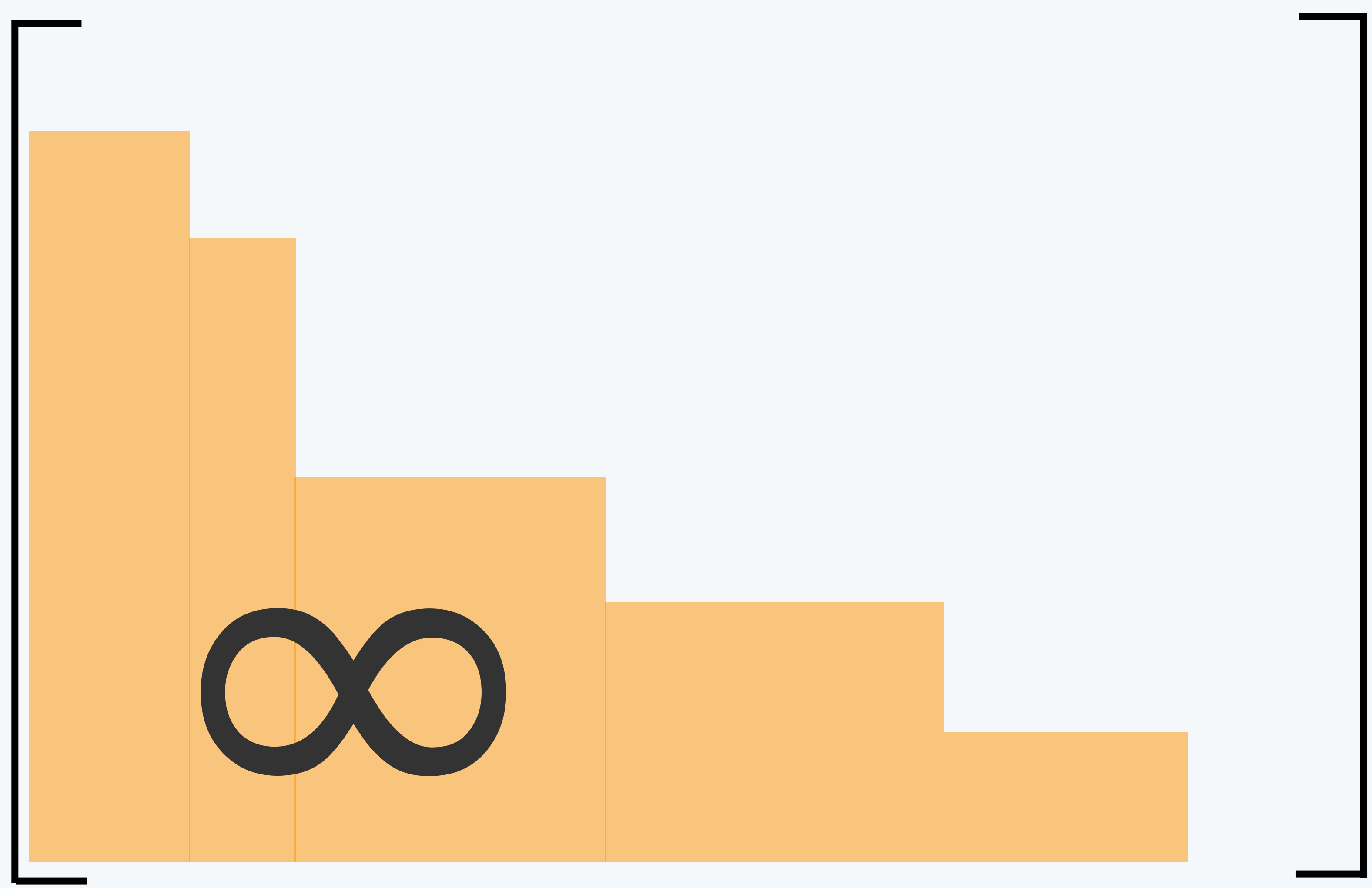
- … but probably not 80+ either.

A Greedy Algorithm

- Given $A \subseteq \mathcal{Y}$, the *induced deterministic mechanism,* $P_{A,}$ is

$$p(y|x) = 1 \quad \text{if} \quad y = \text{argmin}\{c(x, y') : y' \in A\}$$



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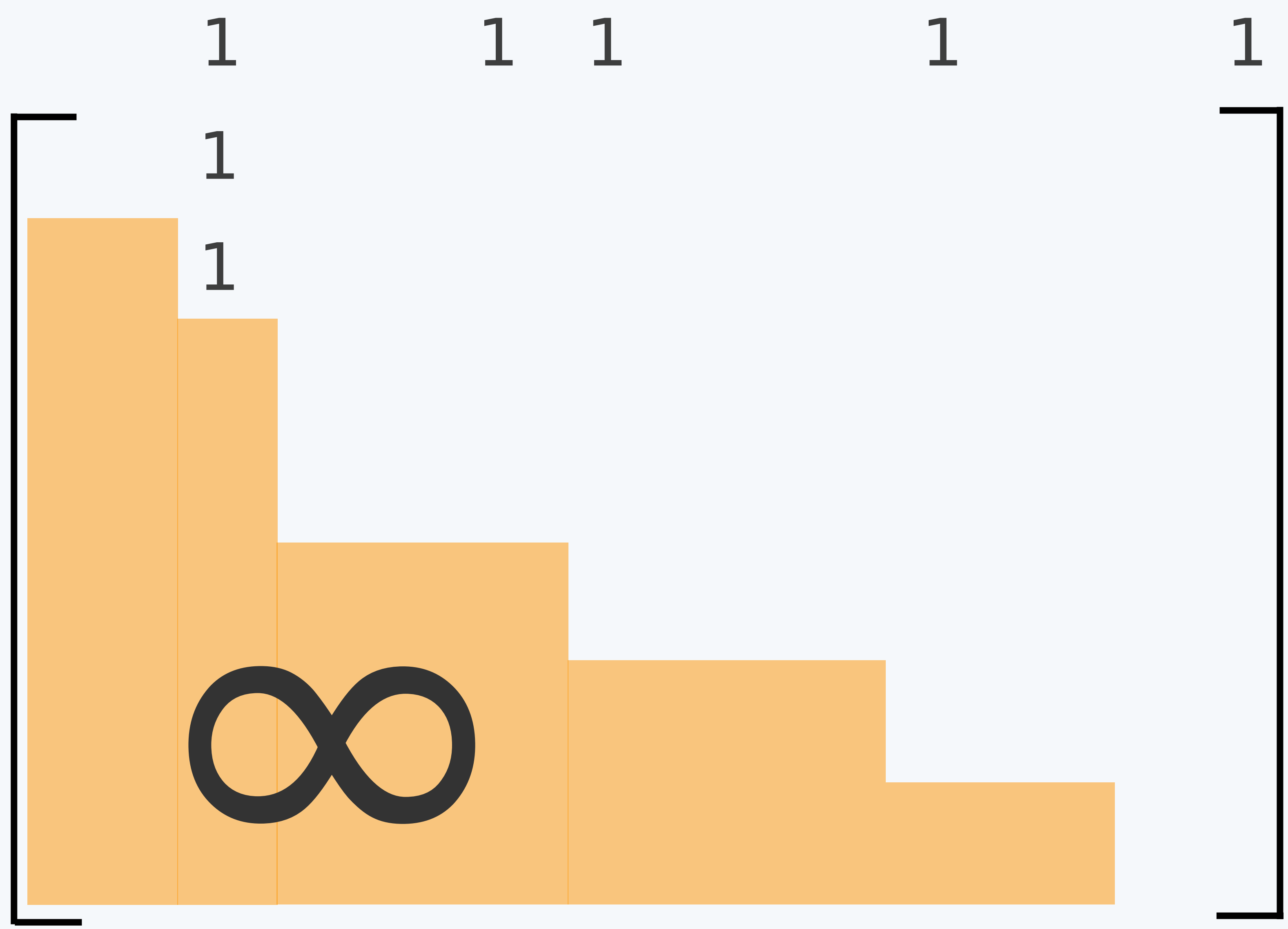

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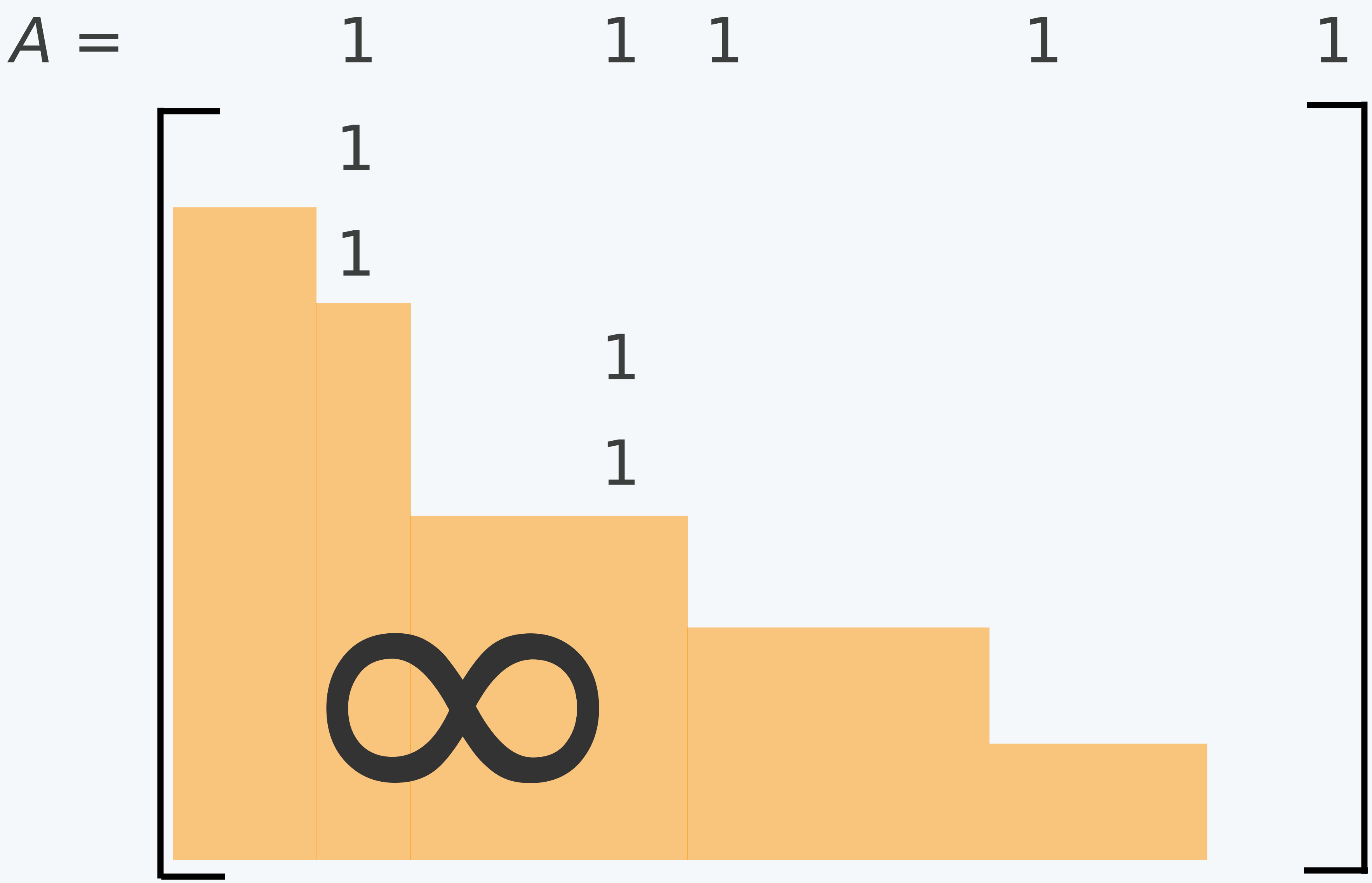

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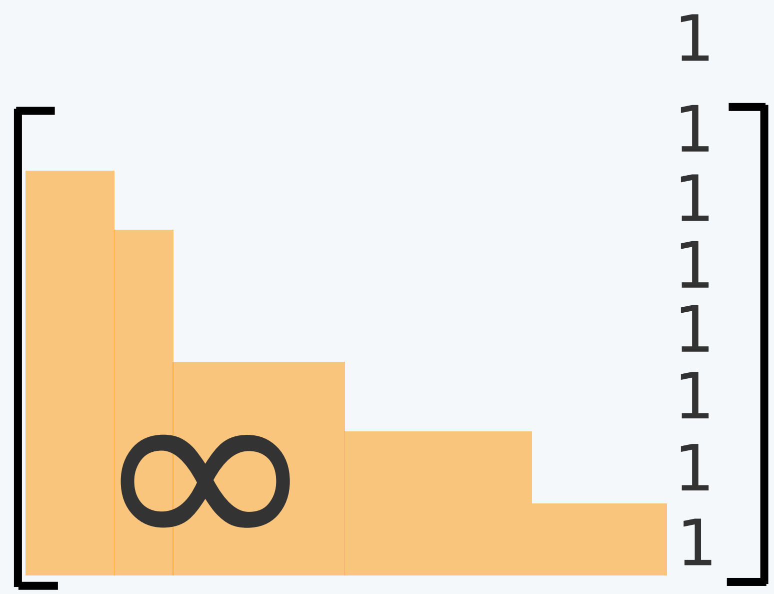

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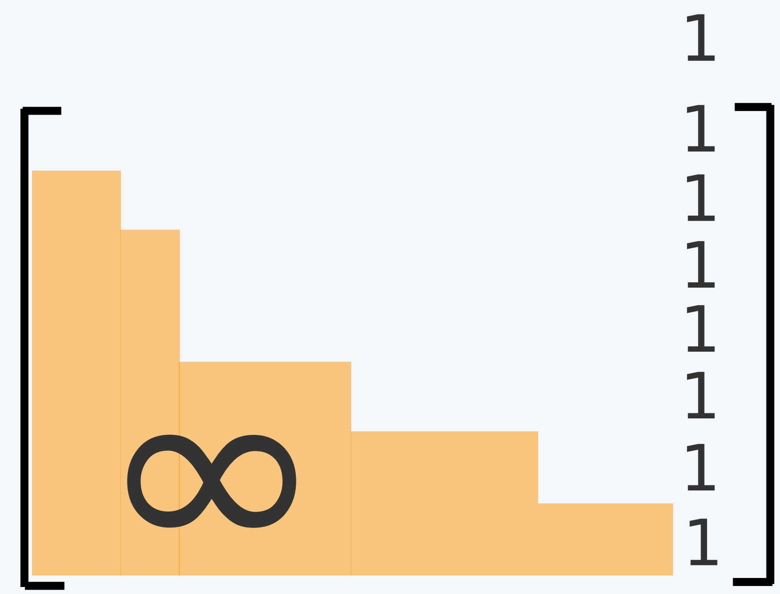
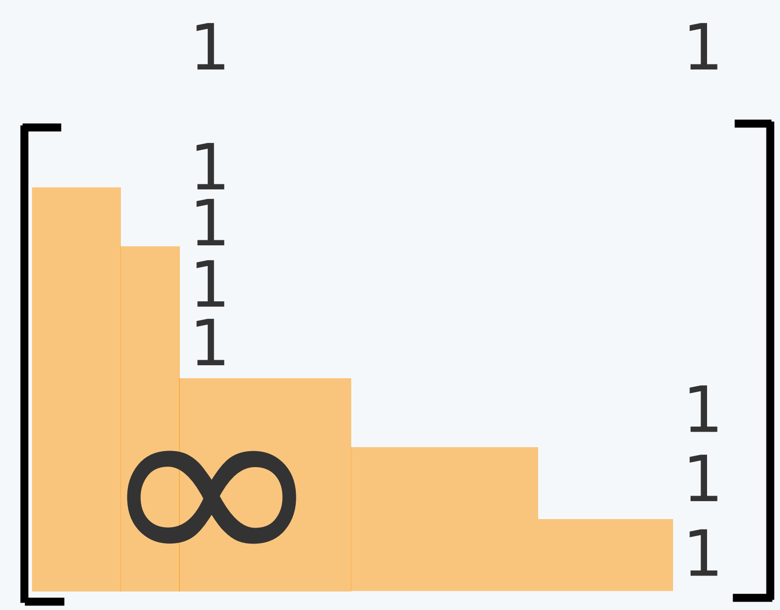


- Iterate: $A \to A \cup \{j\}$, where $j \notin A$ is chosen to minimize the cost of $P_{A \cup \{j\}}$.



Theorem

Theorem (Wu, Wagner, Suh '19):

For exp-leakage k, let

- $C^*(k)$ denote the optimum cost
- $C_\mathrm{G}(k)$ denote the cost obtained by the greedy algorithm

Then $C^*(1) = C_\mathrm{G}(1)$, $C^*(2) = C_\mathrm{G}(2)$, and

$$
\begin{aligned}
C^*(1) - C_G(k) &\geq \left(1 - \left(\frac{k-2}{k-1}\right)^{k-1}\right)(C^*(1) - C^*(k)) \\
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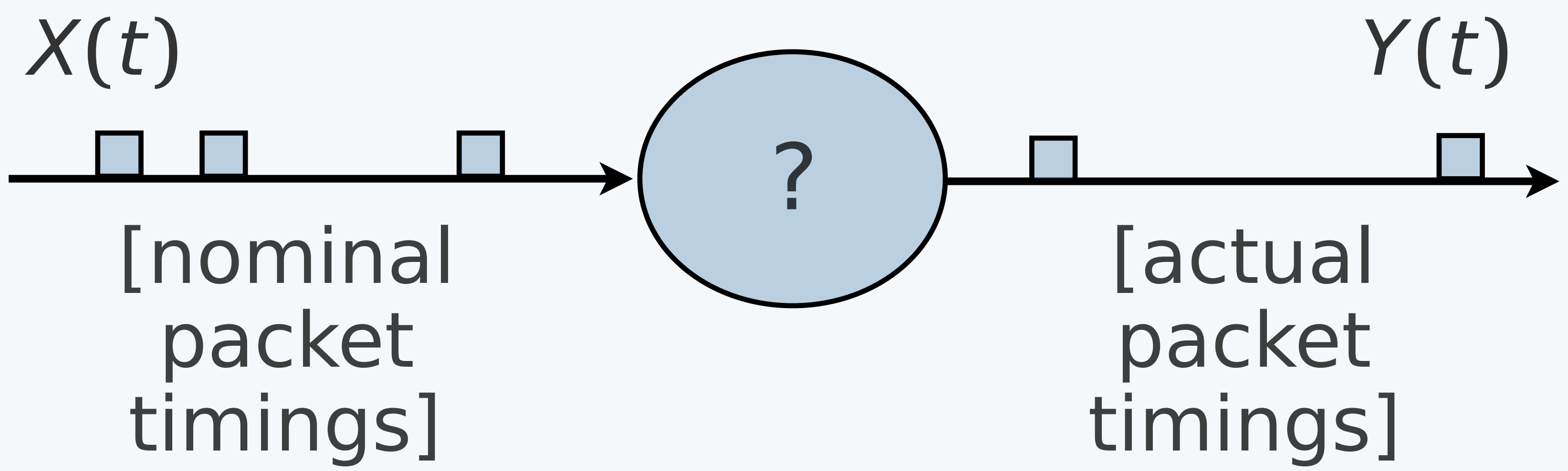
Note: leads to a sequence of approximations.

How to Delay Packets?



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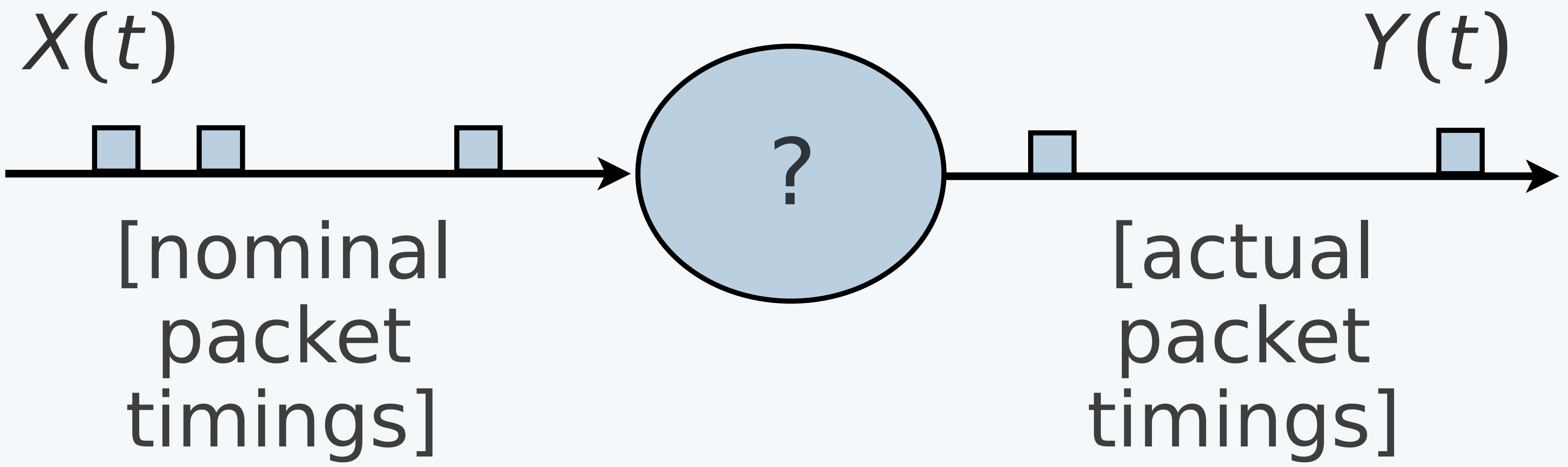

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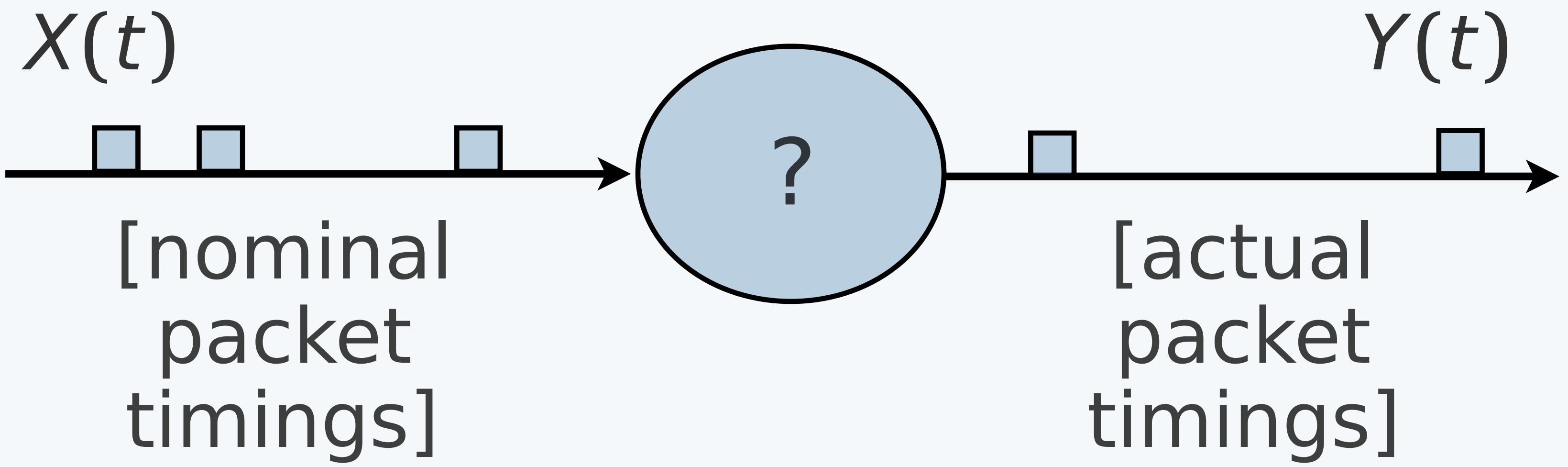
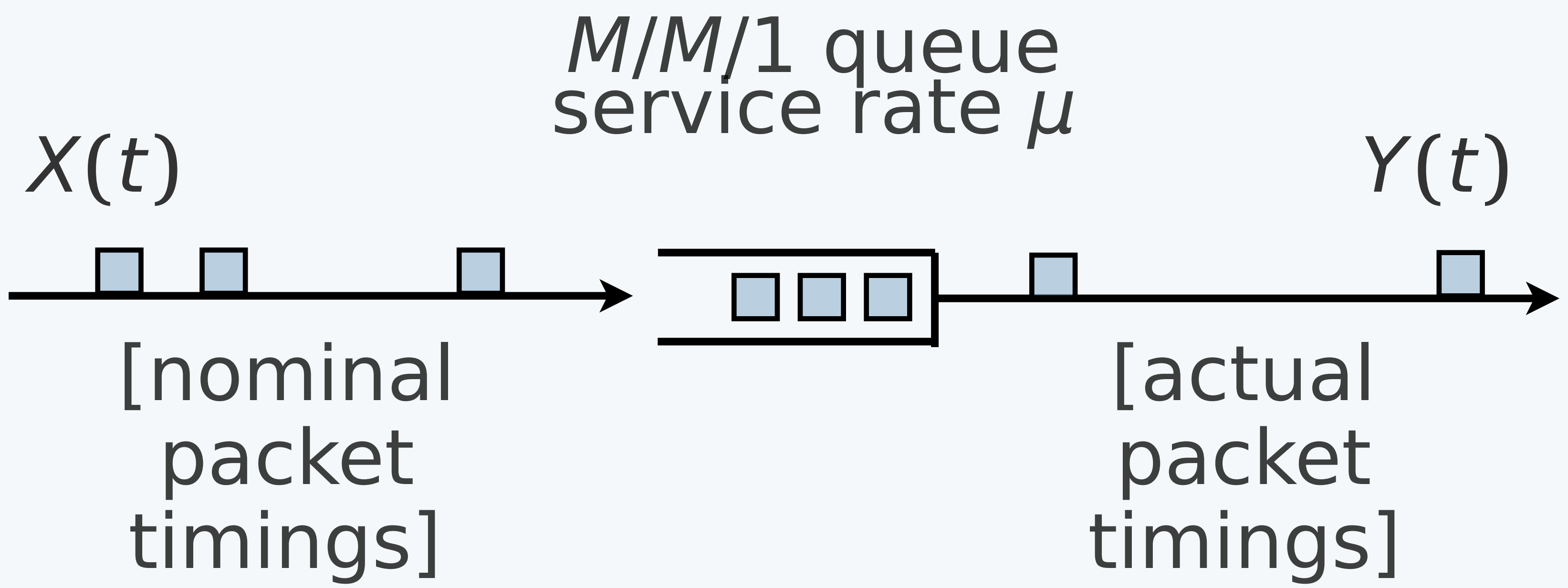


- Suppose $X(t)$ is a Poisson process with rate λ
- How to blur the packet timings to minimize leakage?

Try an $M/M/1$ Queue



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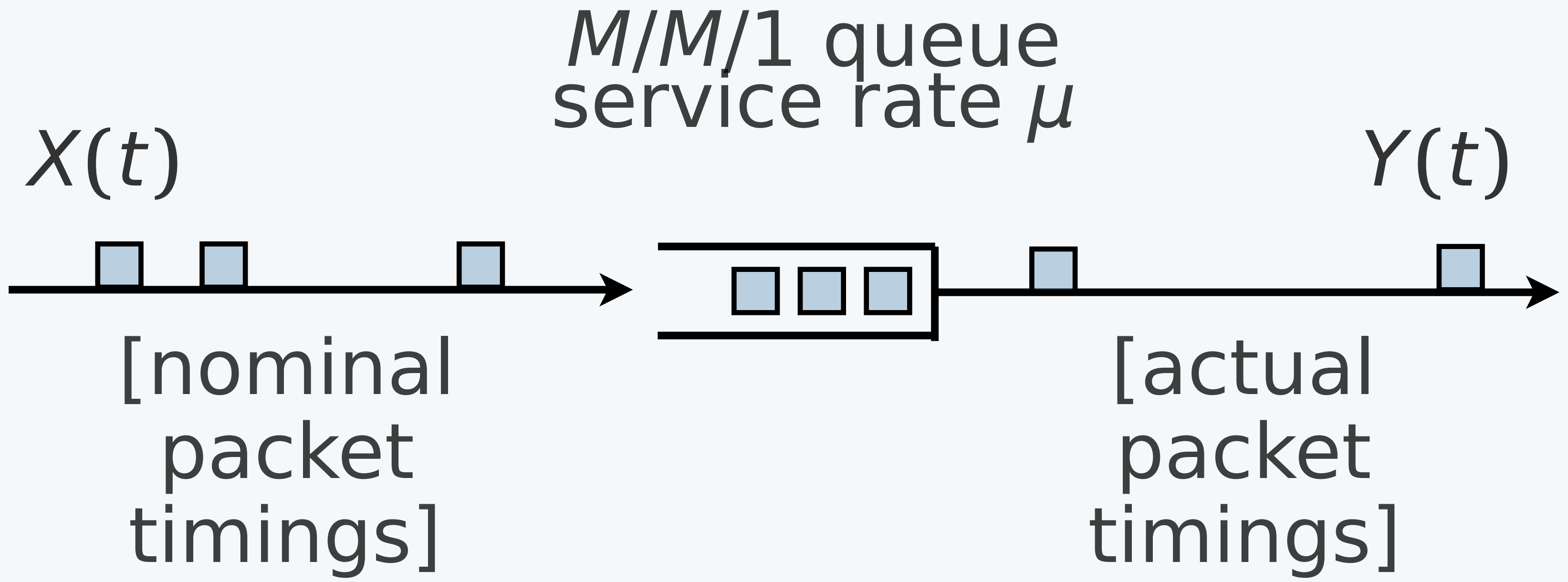

$$\frac{1}{T} \cdot \mathcal{L}\left(\{X(t)\}_{t=0}^{T} \to \{Y(t)\}_{t=0}^{T}\right) = \mu \quad \text{nats}$$

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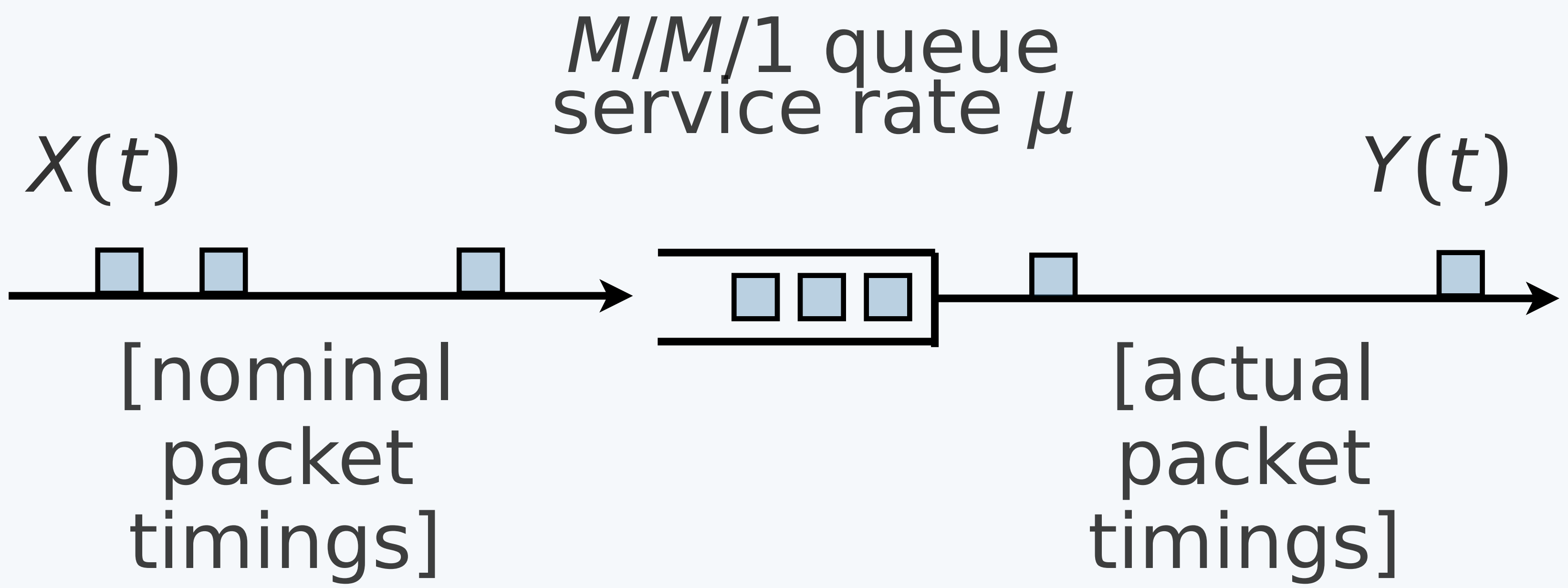

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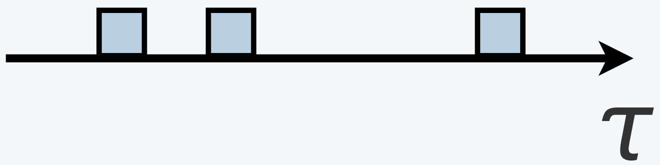
[leakage rate is at least λ]

Accumulate and Dump



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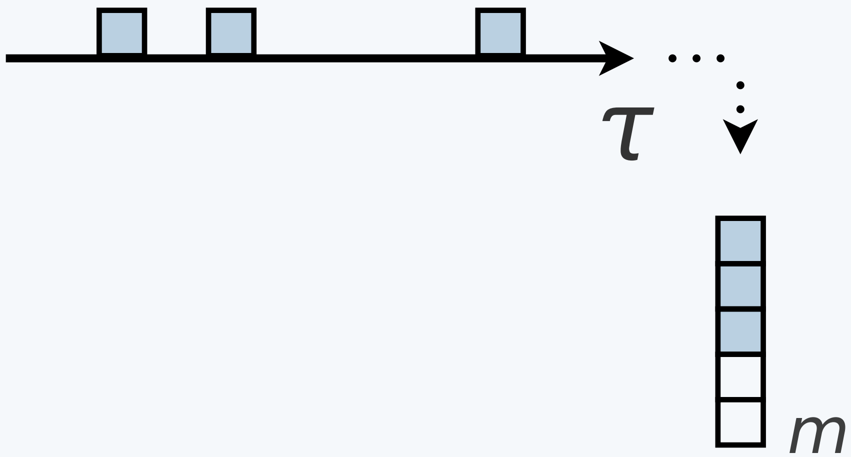

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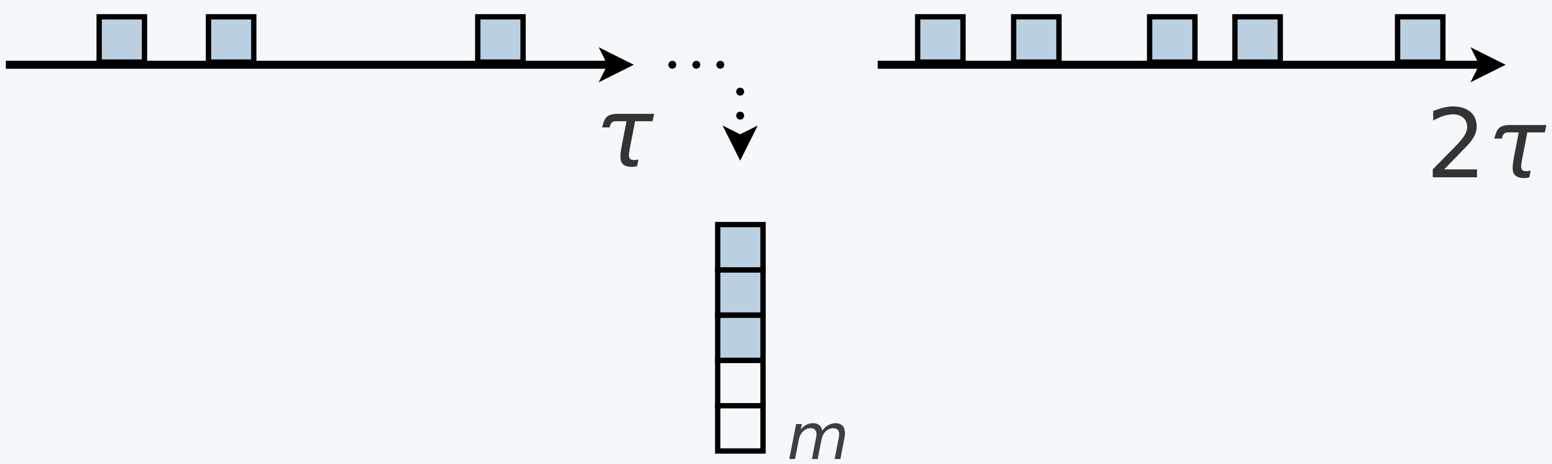

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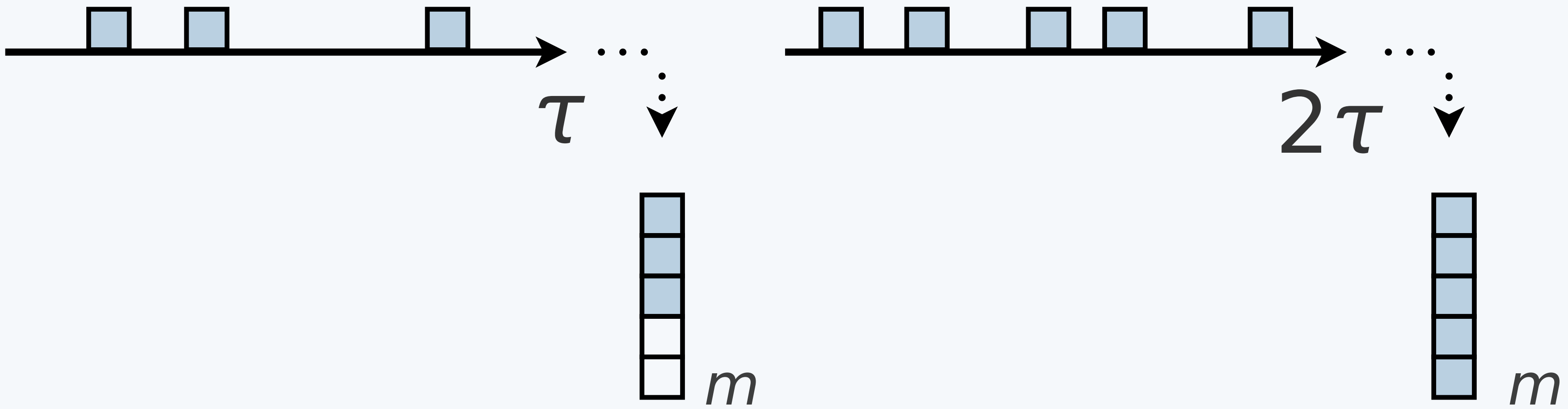

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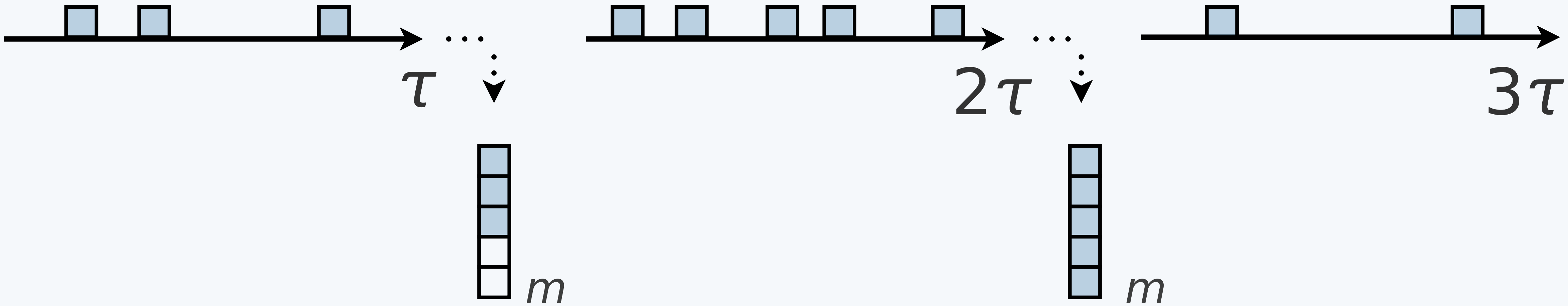

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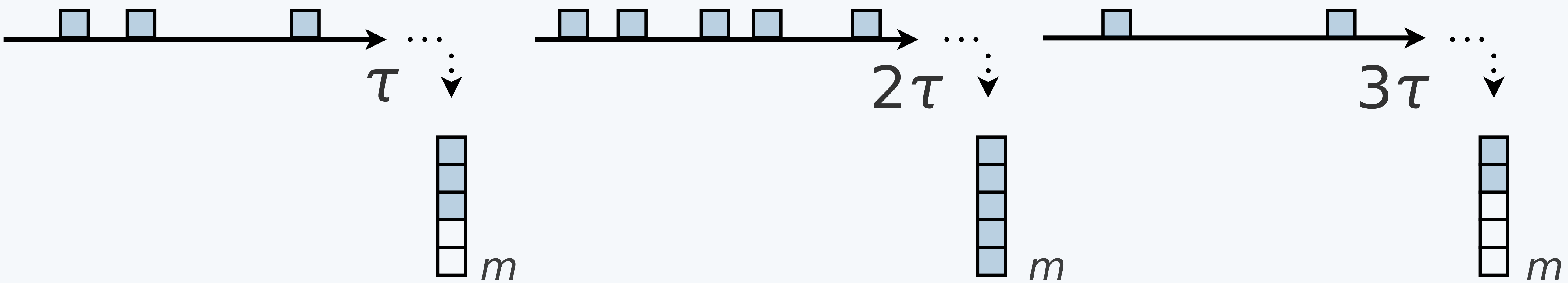

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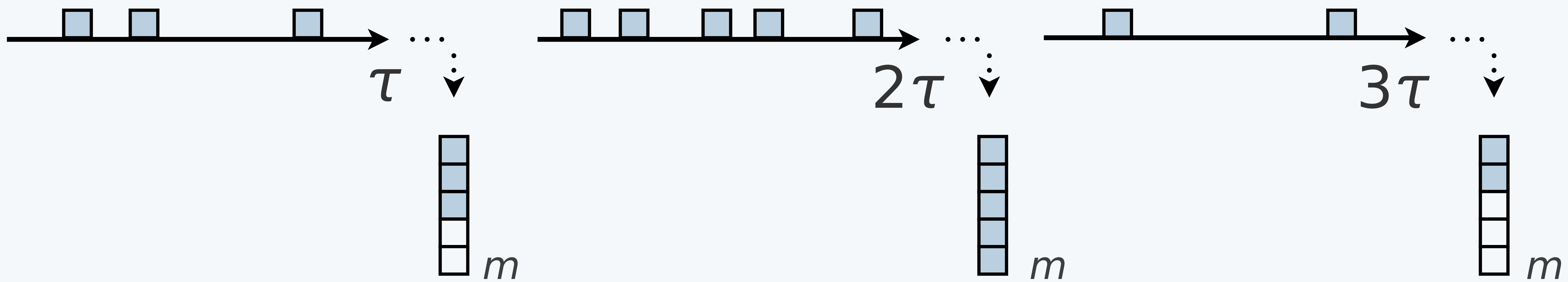

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[quantization leaks less than “adding noise”]

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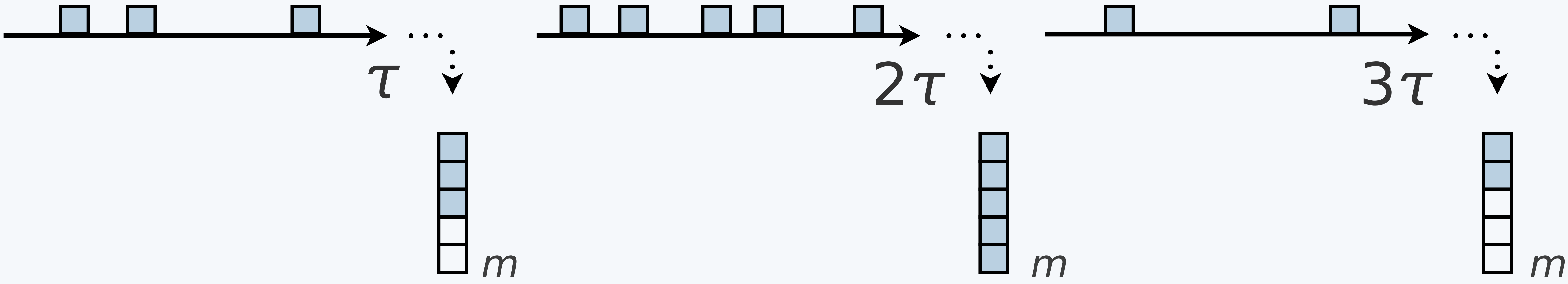

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[cf. Kadloor, Kiyavash, and Venkitasubramaniam '16]

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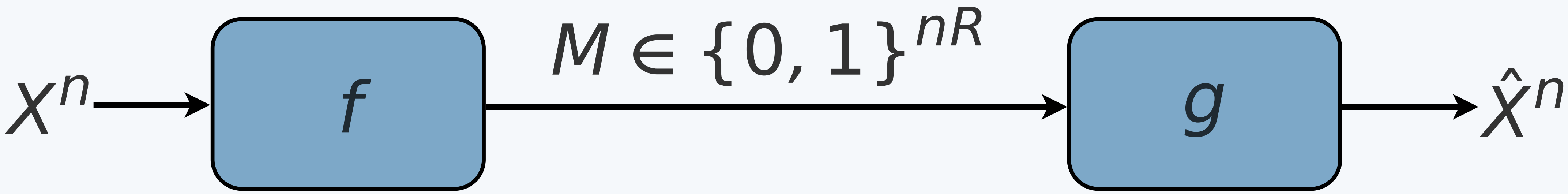

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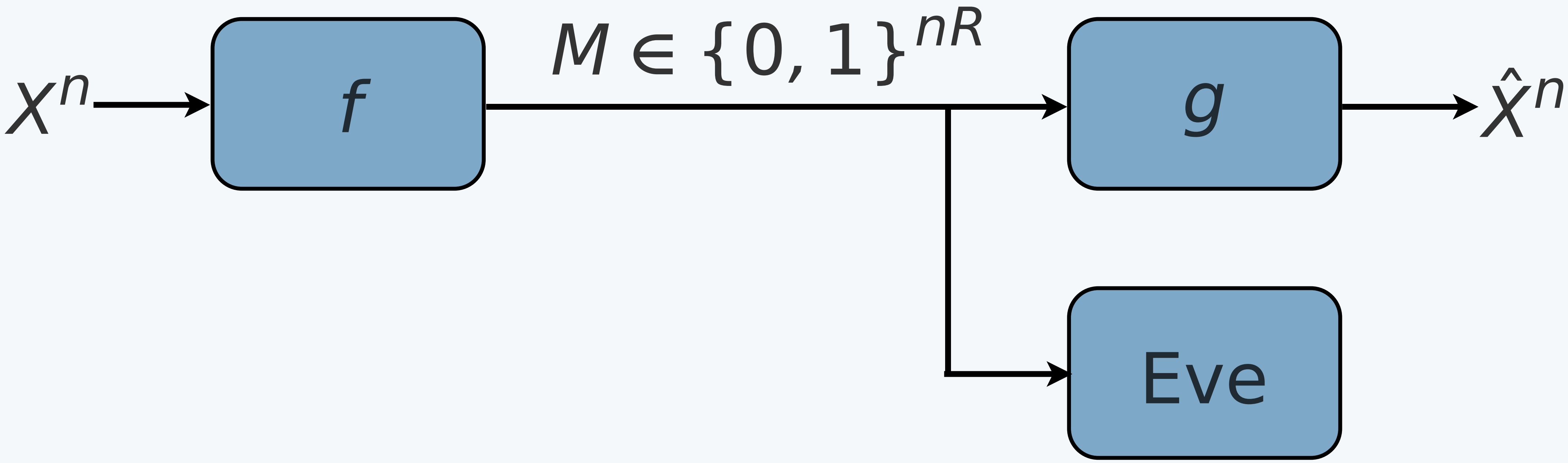

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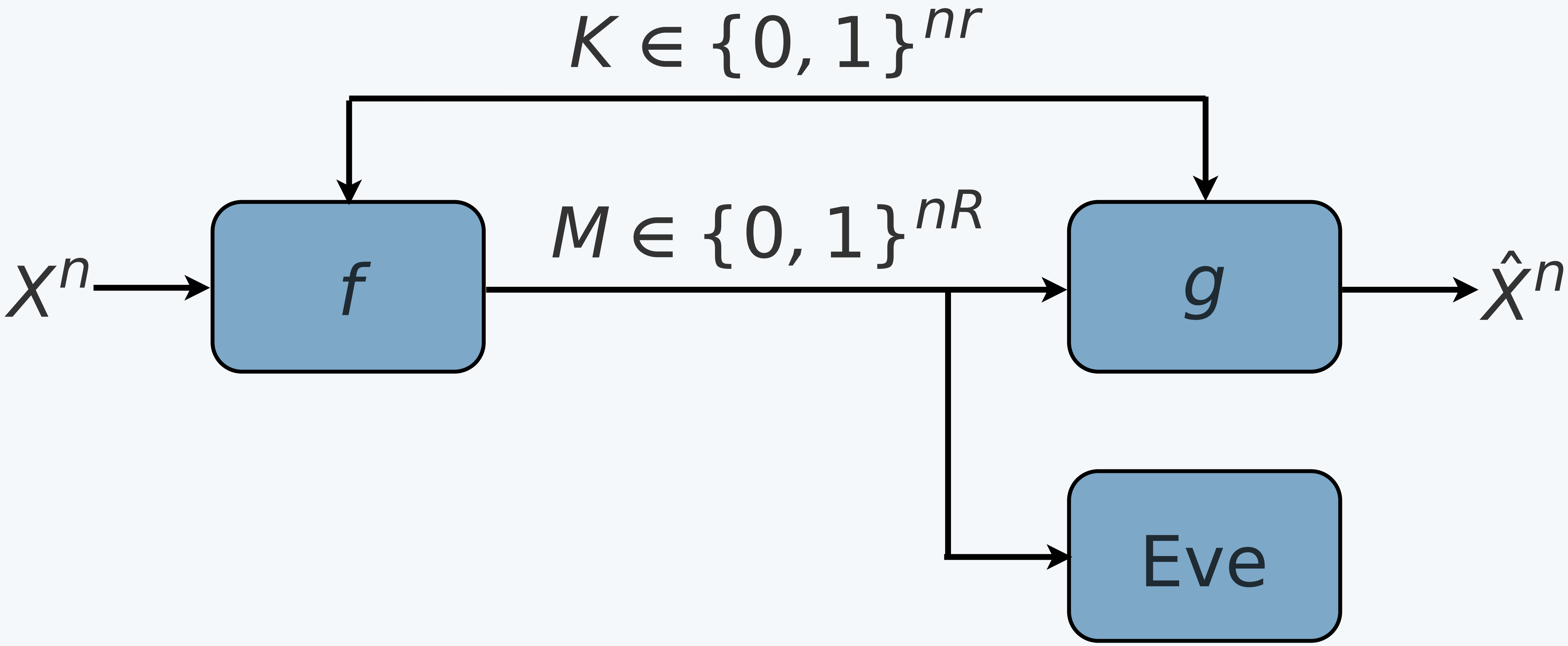
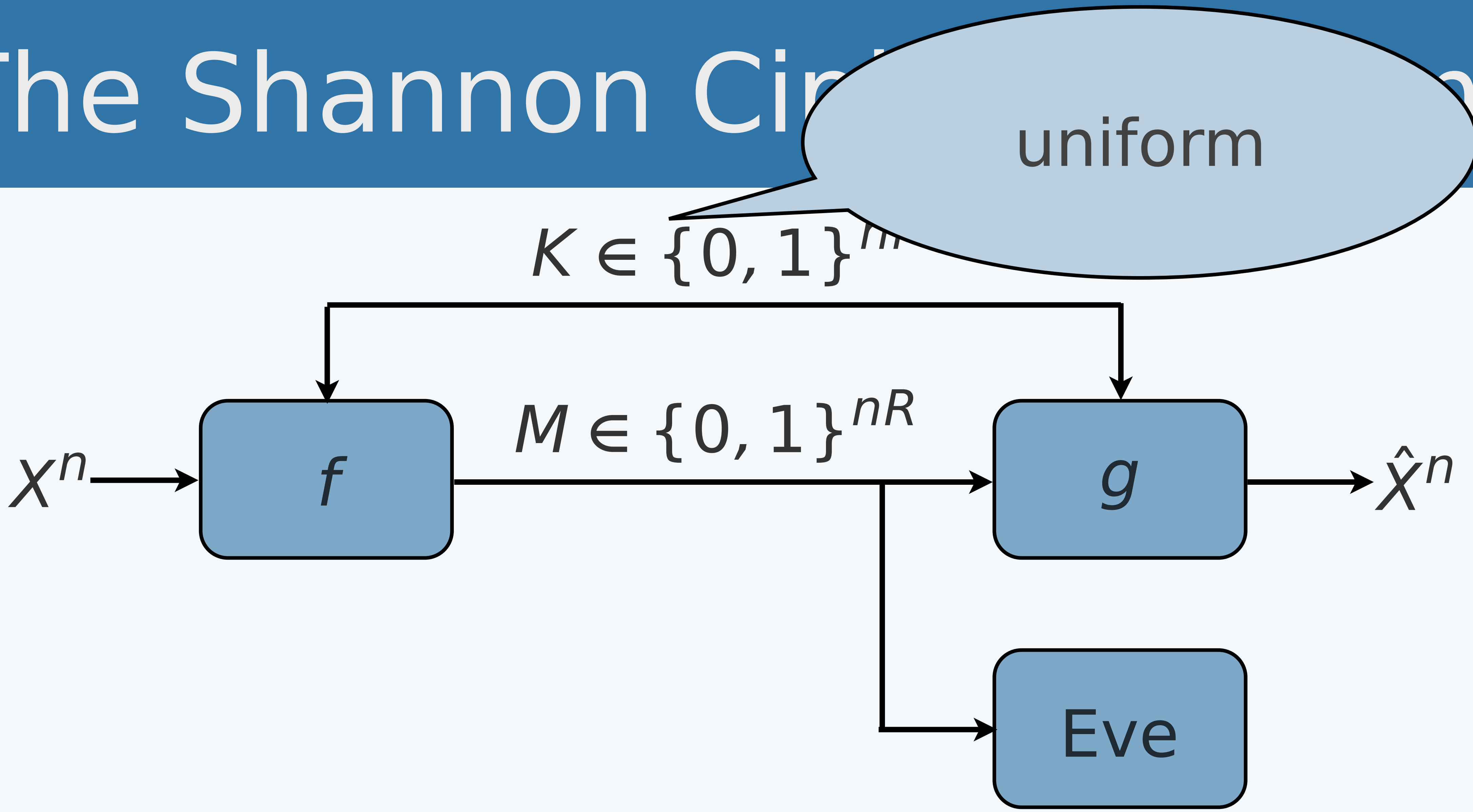
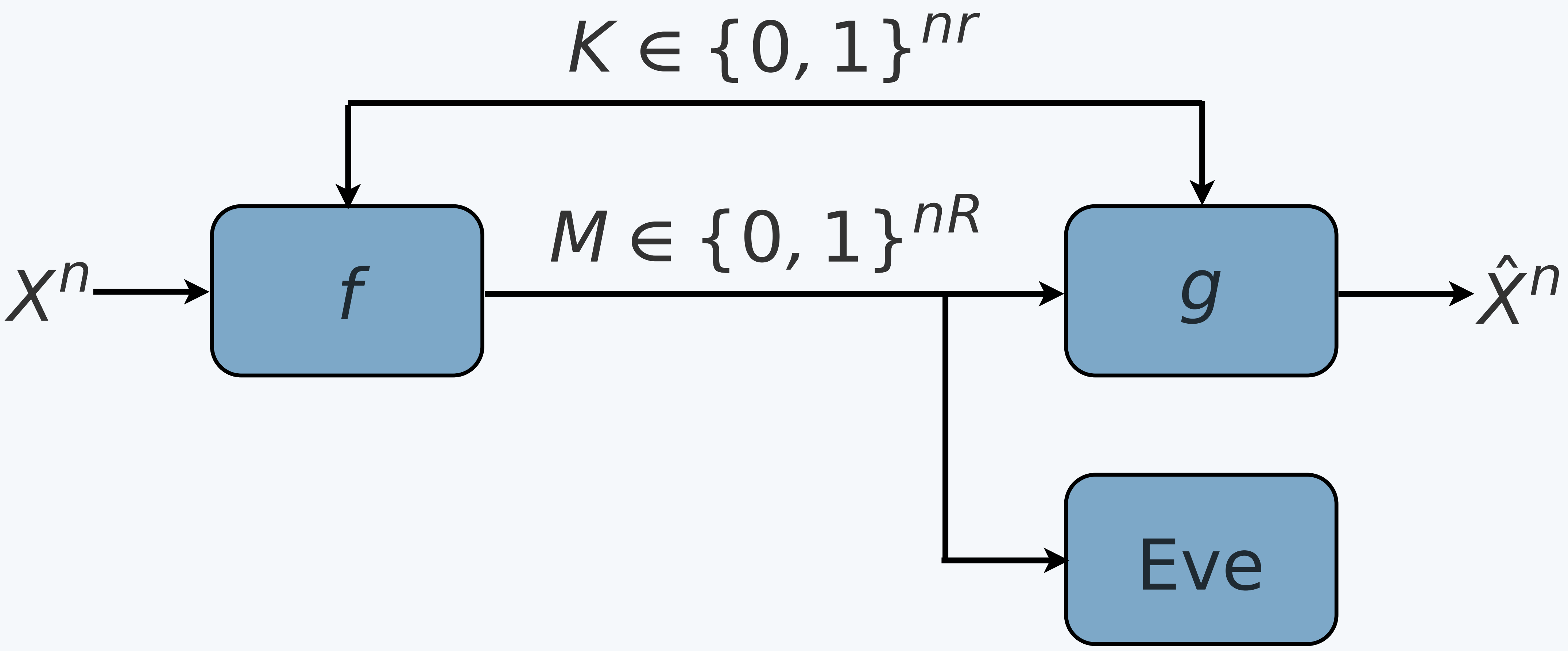


The Shannon Cip[...]



The Shannon Cipher System



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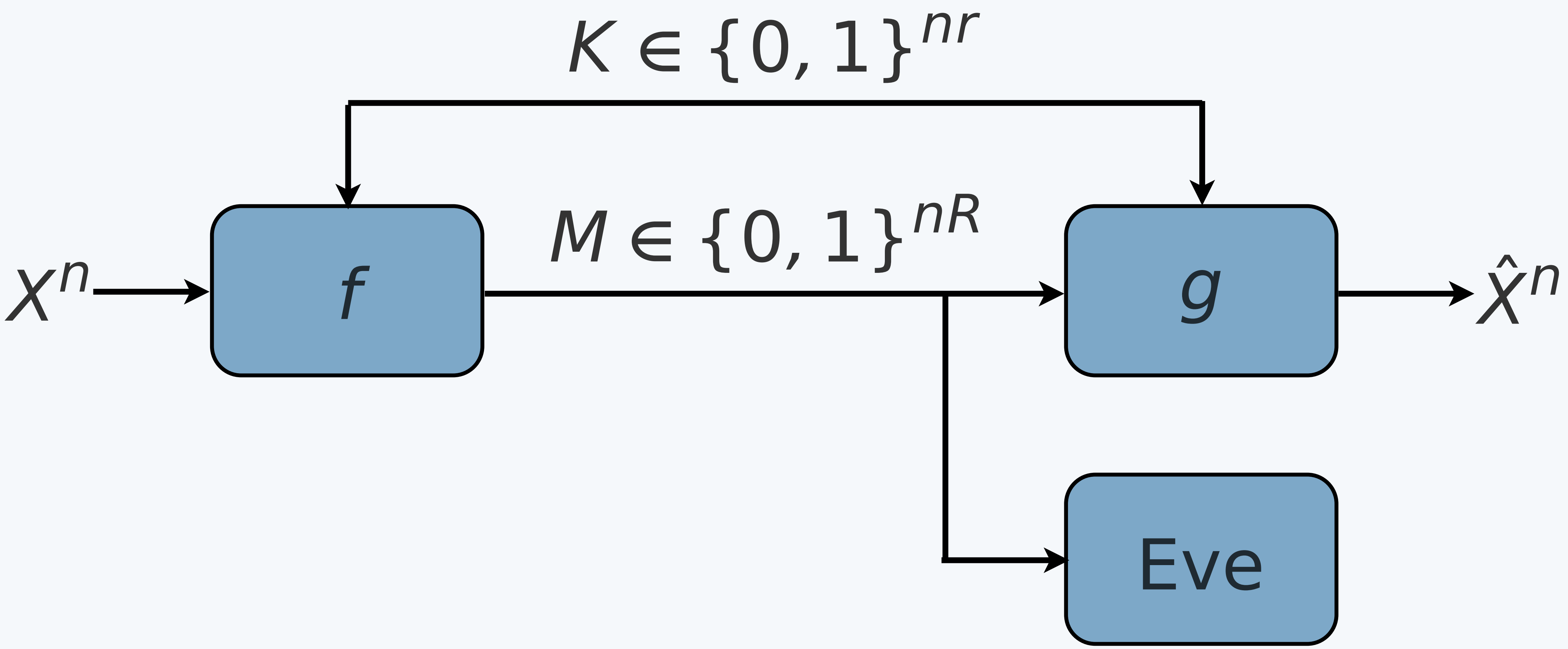

- Shannon ('49): perfect secrecy is possible if(f) the key rate r exceeds the message rate R.

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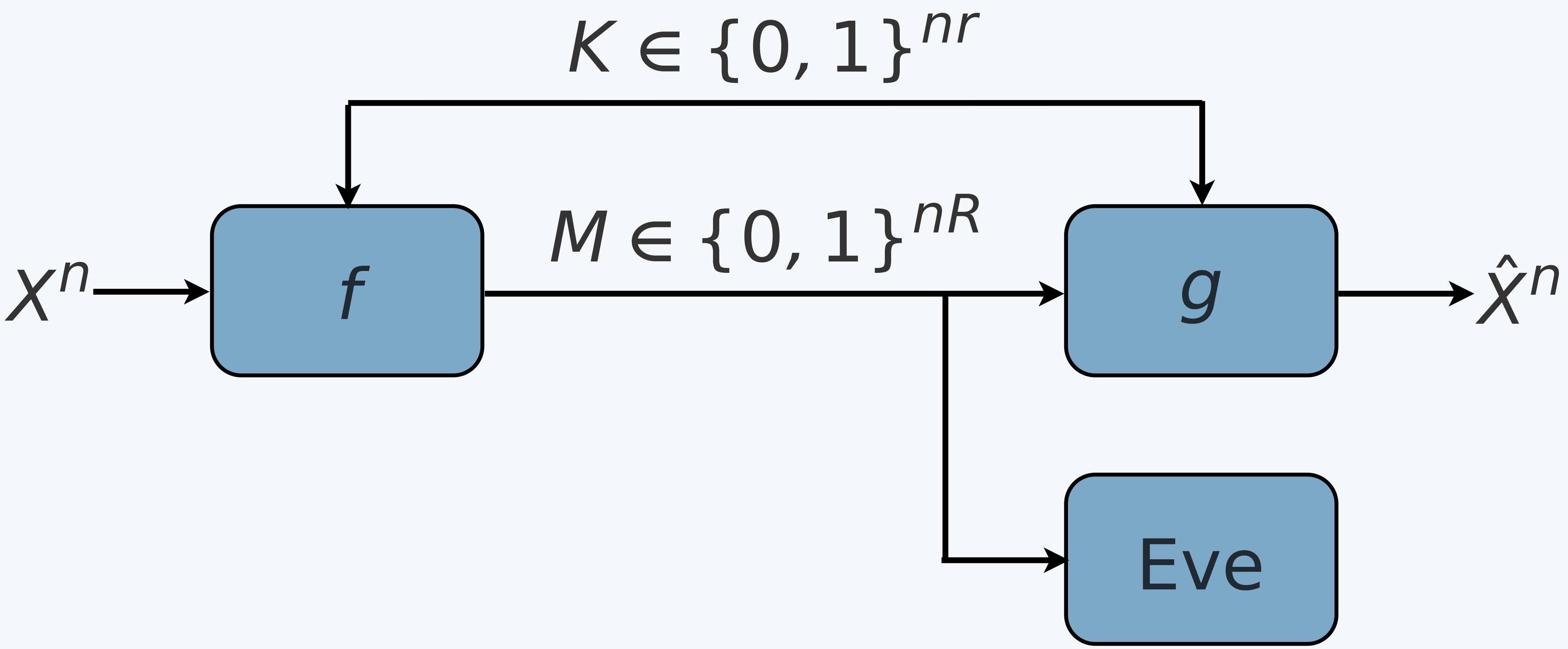
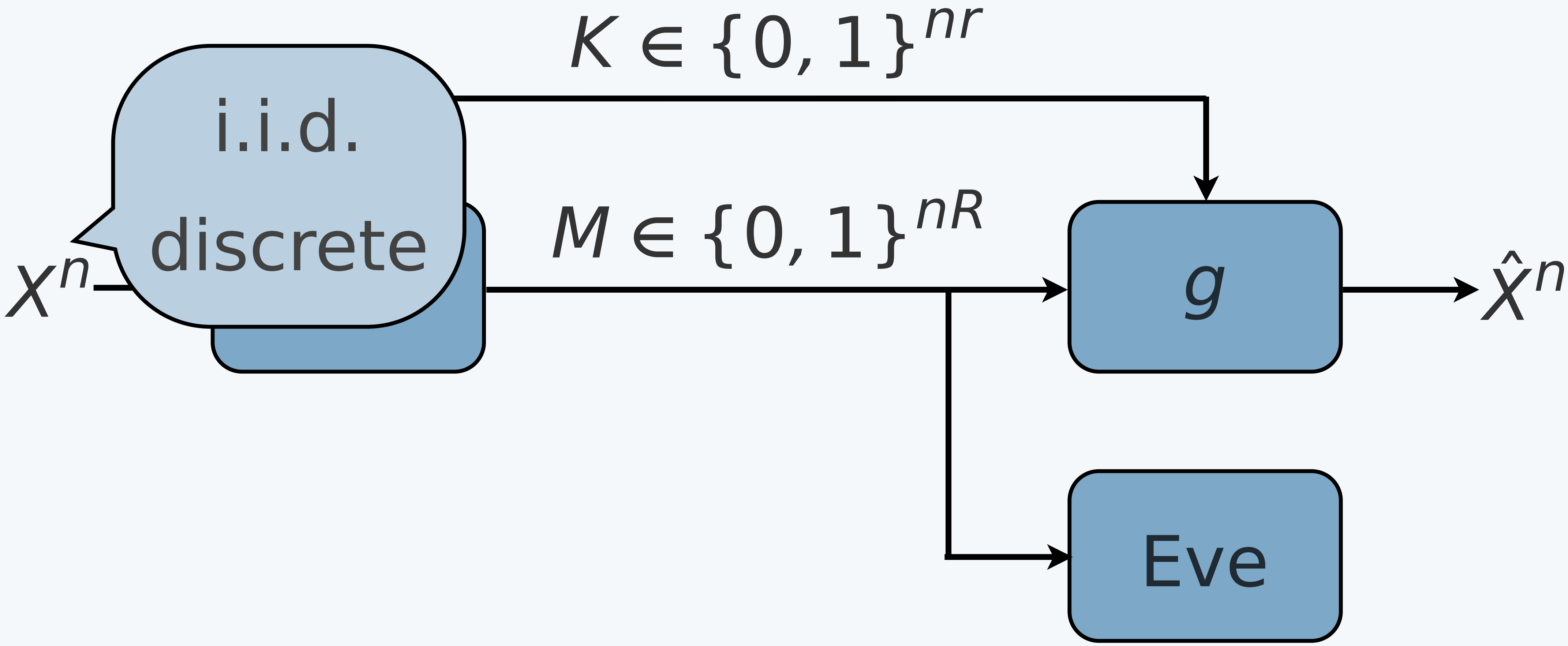


- Shannon ('49): perfect secrecy is possible if(f) the key rate r exceeds the message rate R.
- How to design f and g to minimize leakage when $r < R$?

Leakage and Shannon's Cipher



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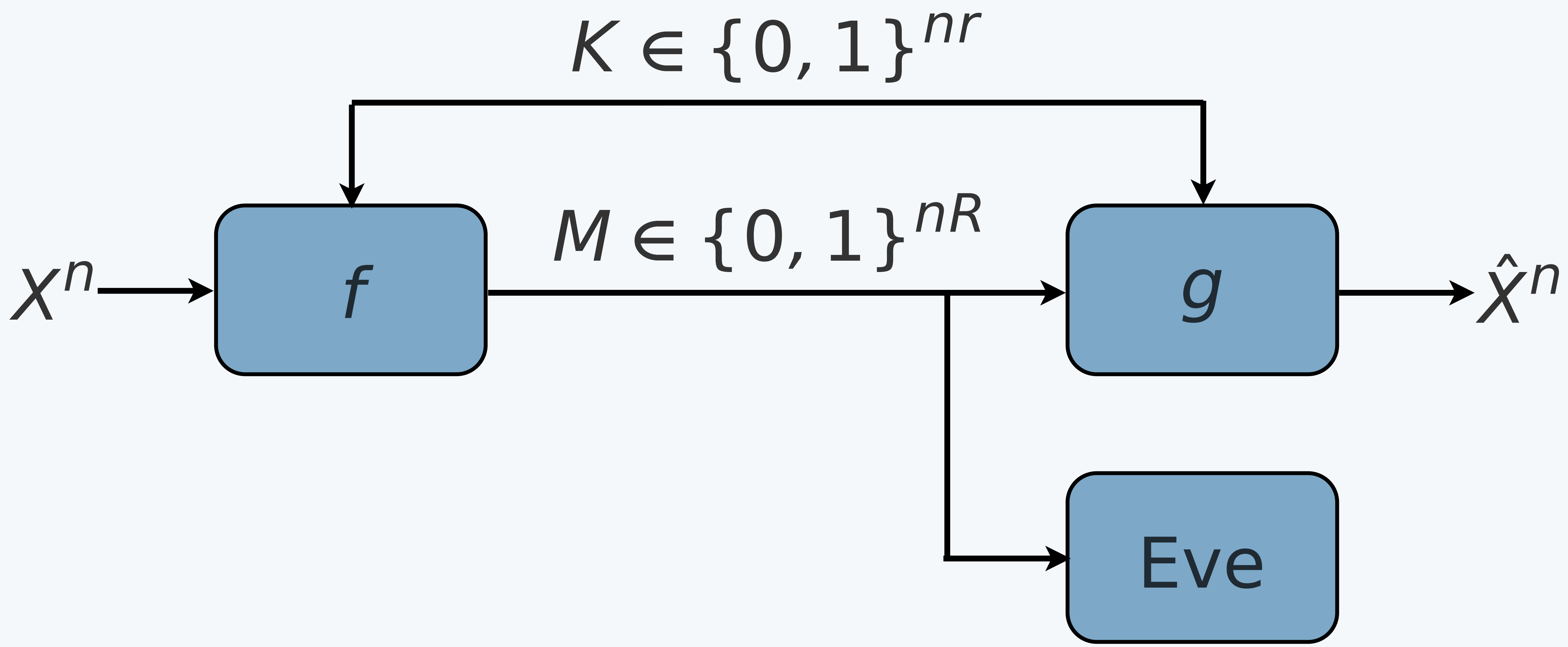

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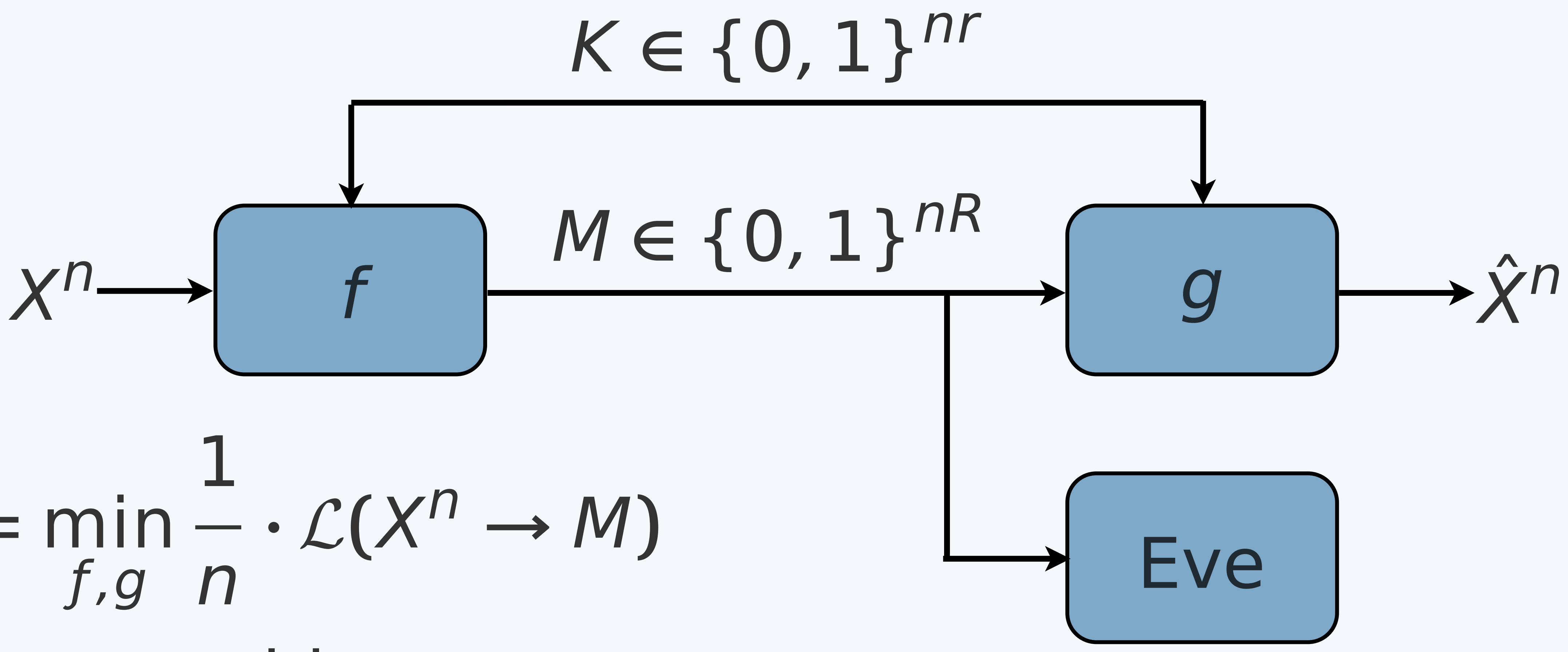

$$L_n = \min_{f,g} \frac{1}{n} \cdot \mathcal{L}(X^n \to M)$$

subject to

$$f : \mathcal{X}^n \times \{0,1\}^{nr} \mapsto \{0,1\}^{nR}$$

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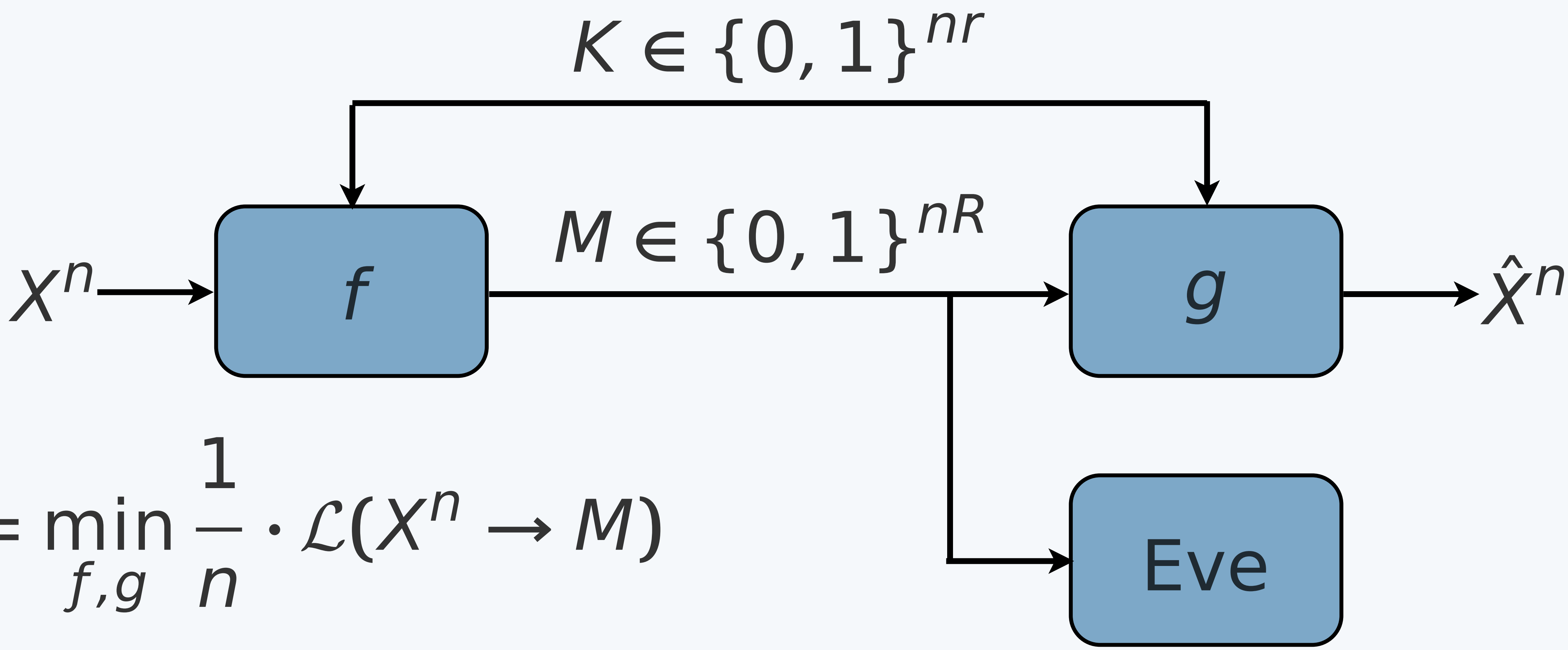
Leakage and Shannon's Cipher



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Leakage and Shannon's Cipher

Theorem (Issa-Kamath-Wagner): Let $R(D)$ denote the rate-distortion function for the source. If

$$R < R(D),$$

then the problem is infeasible. Otherwise, the min. max. leakage is

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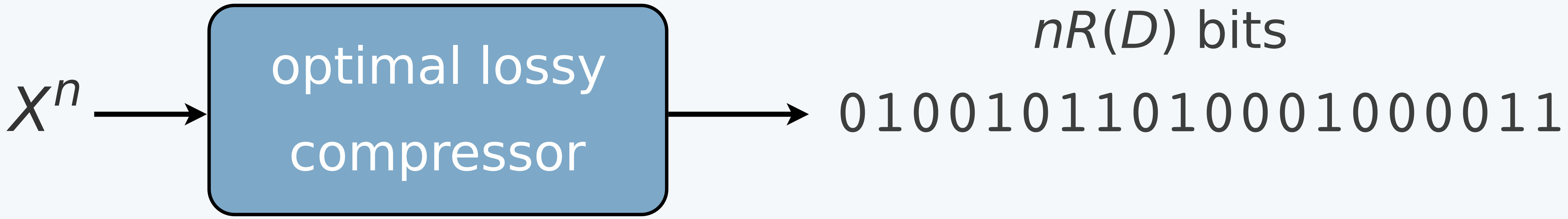
Achievability for Primary User

$X^n \longrightarrow$

Achievability for Primary User



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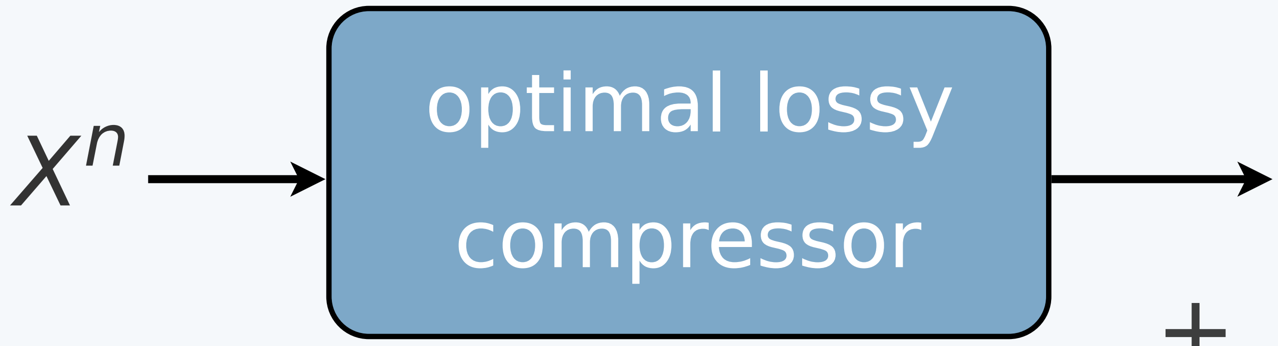
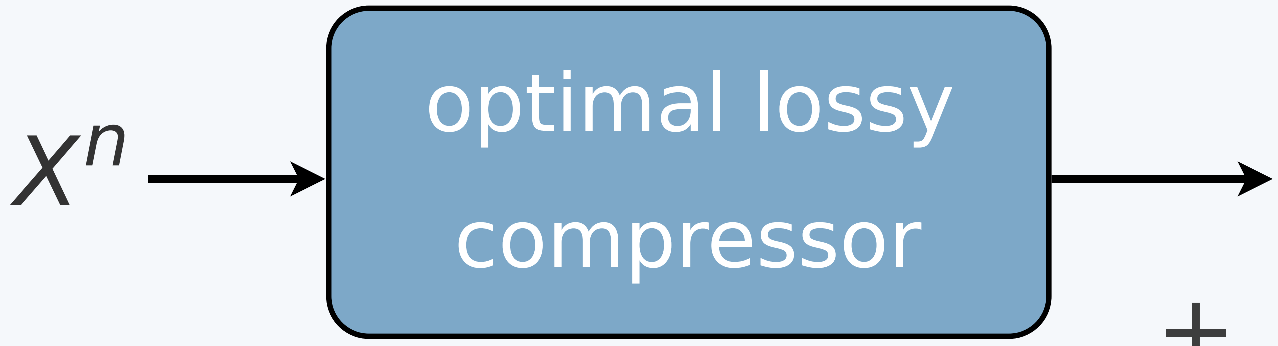


$nR(D)$ bits

0 1 0 0 1 0 1 1 0 1 0 0 0 1 0 0 0 0 1 1

\+ 1 1 0 1 0 1 1 1 0 0 0 (key)

Achievability for Primary User



$nR(D)$ bits

010010111010001000011

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10011100010001000011

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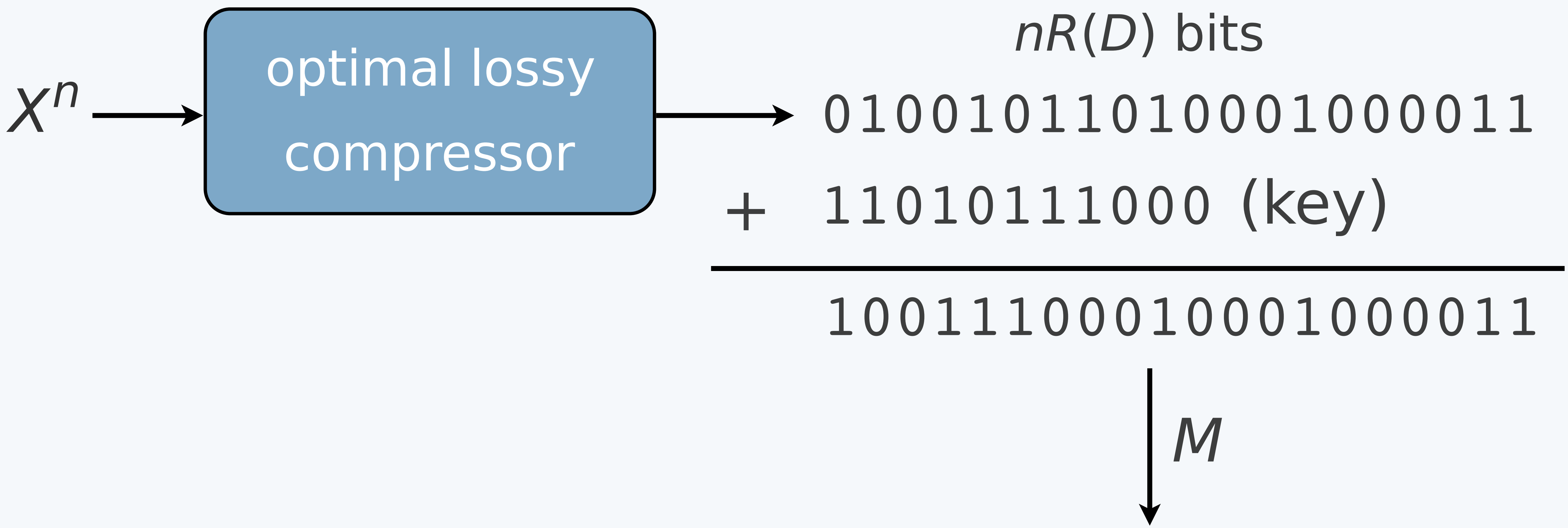

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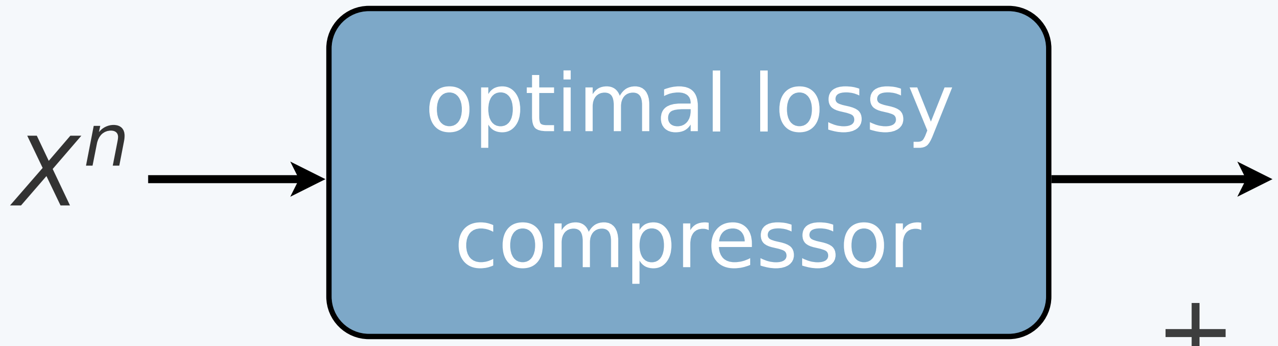


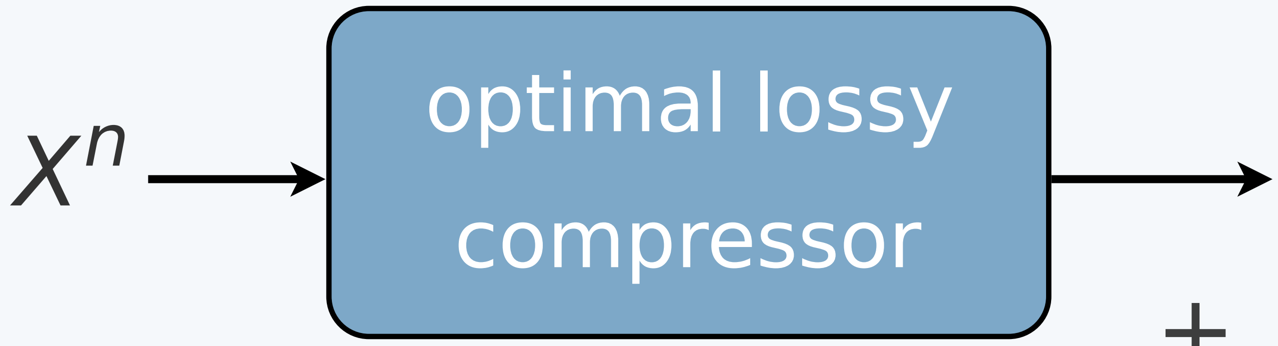
$nR(D)$ bits

$$
\begin{array}{rl}
 & 01001011010001000011 \\
+ & 11010111000 \text{ (key)} \\
\hline
 & 10011100010001000011
\end{array}
$$

$\Big\downarrow M$

10011100010001000011

Achievability for Primary User



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```
  0100101101000100011
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---------------------
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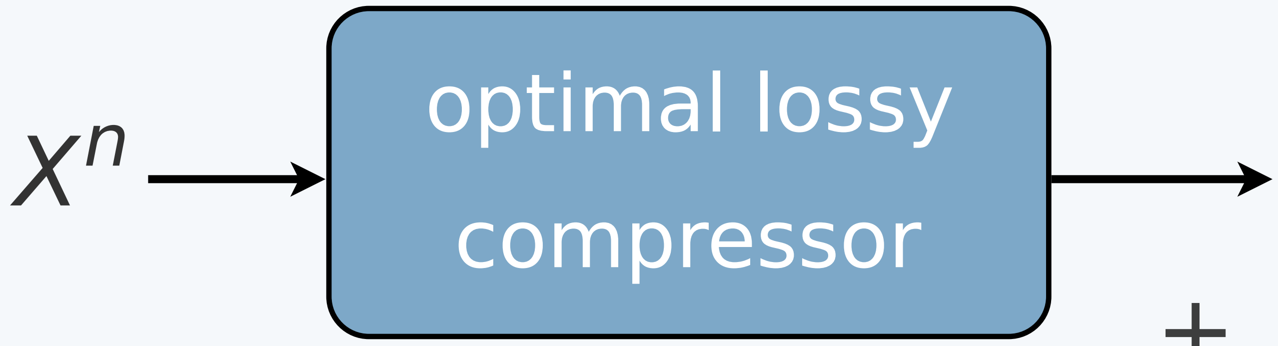

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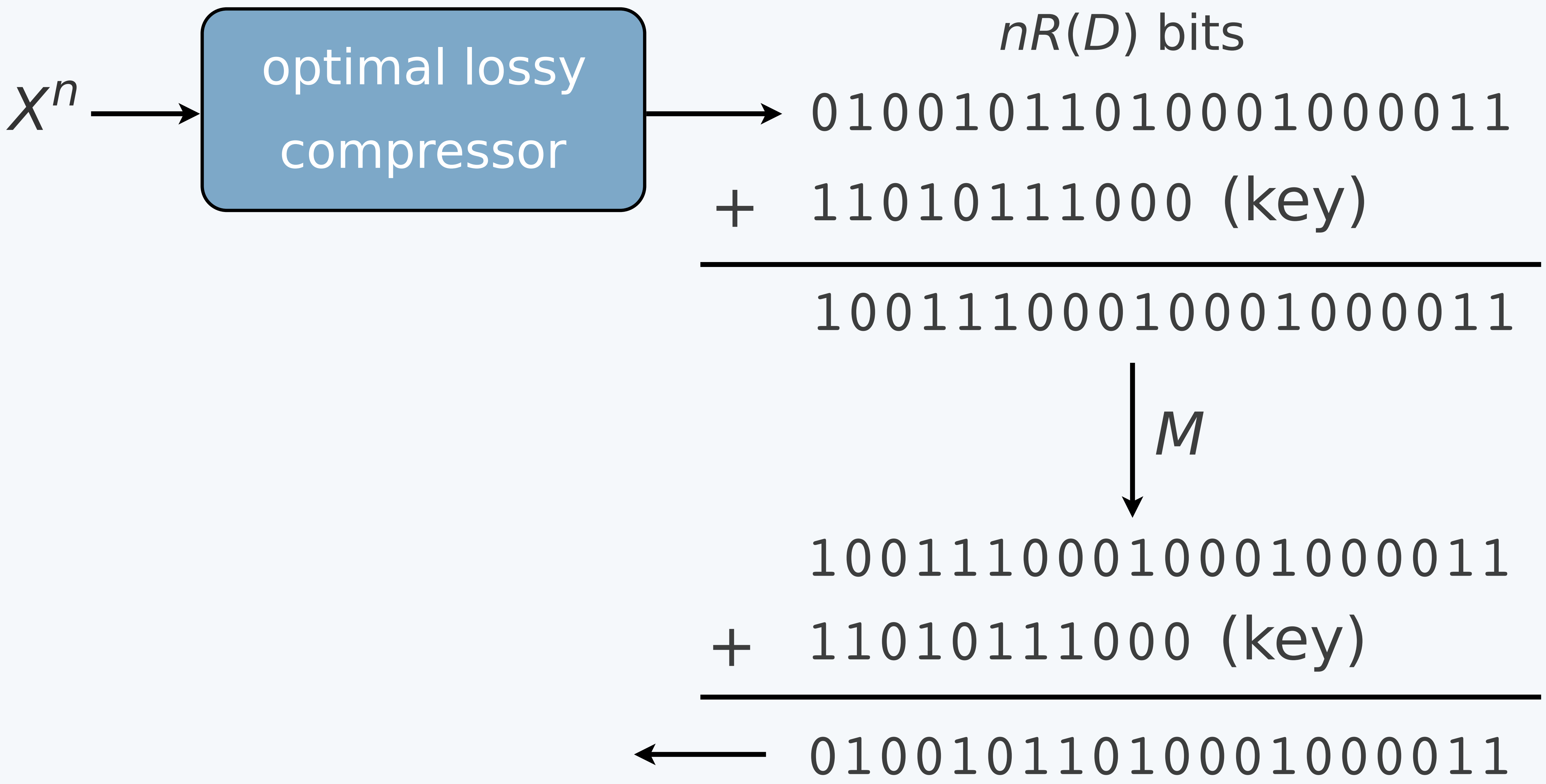
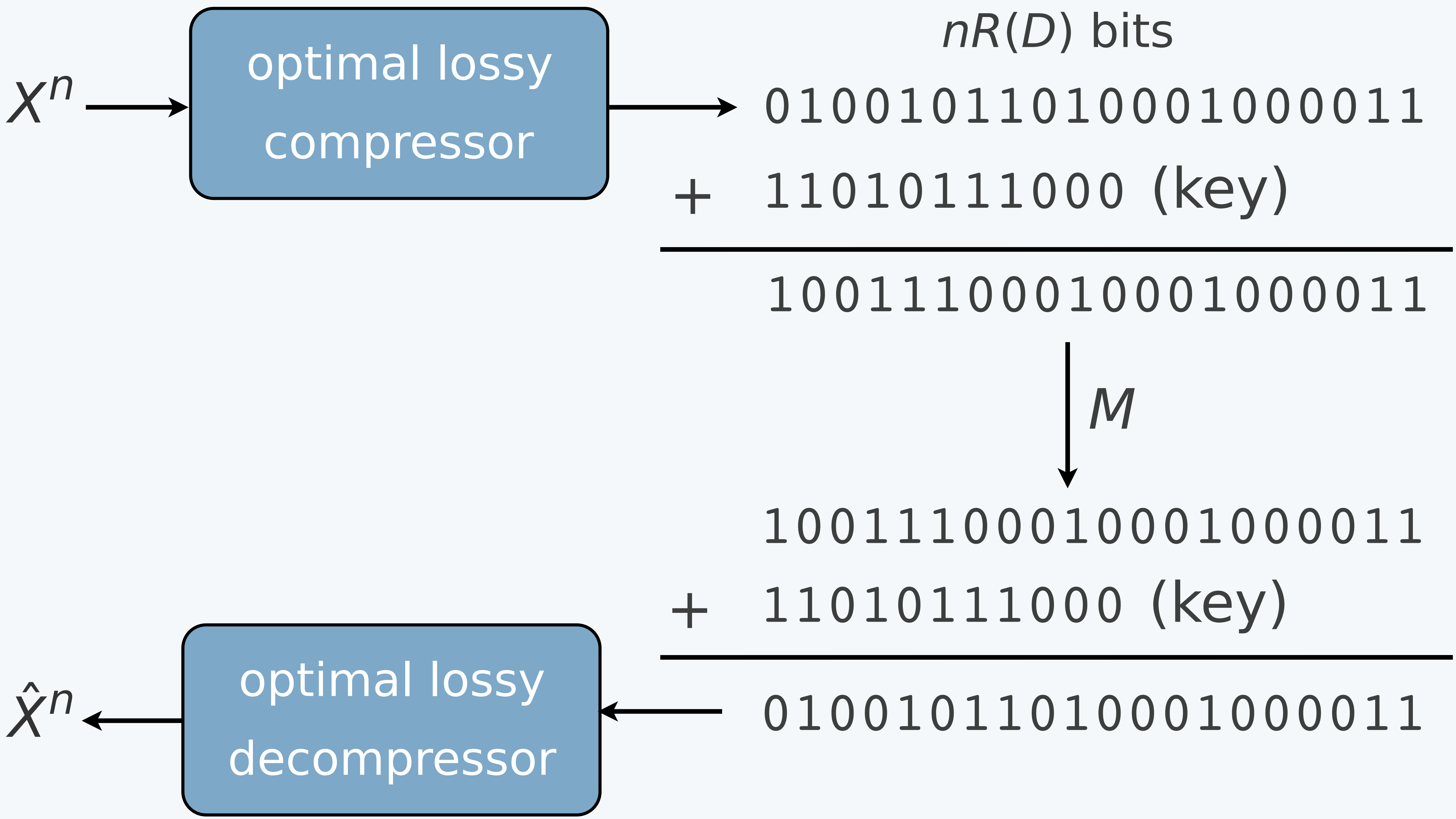
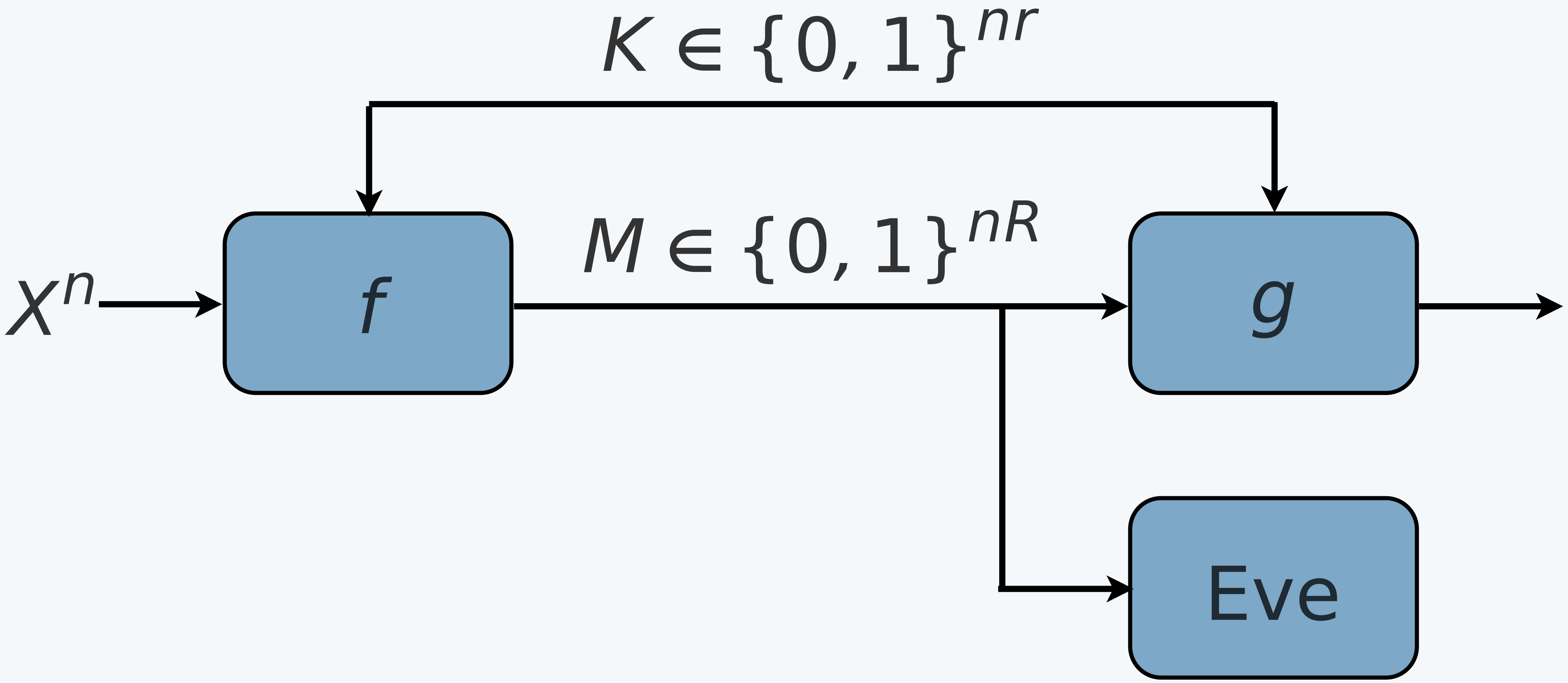


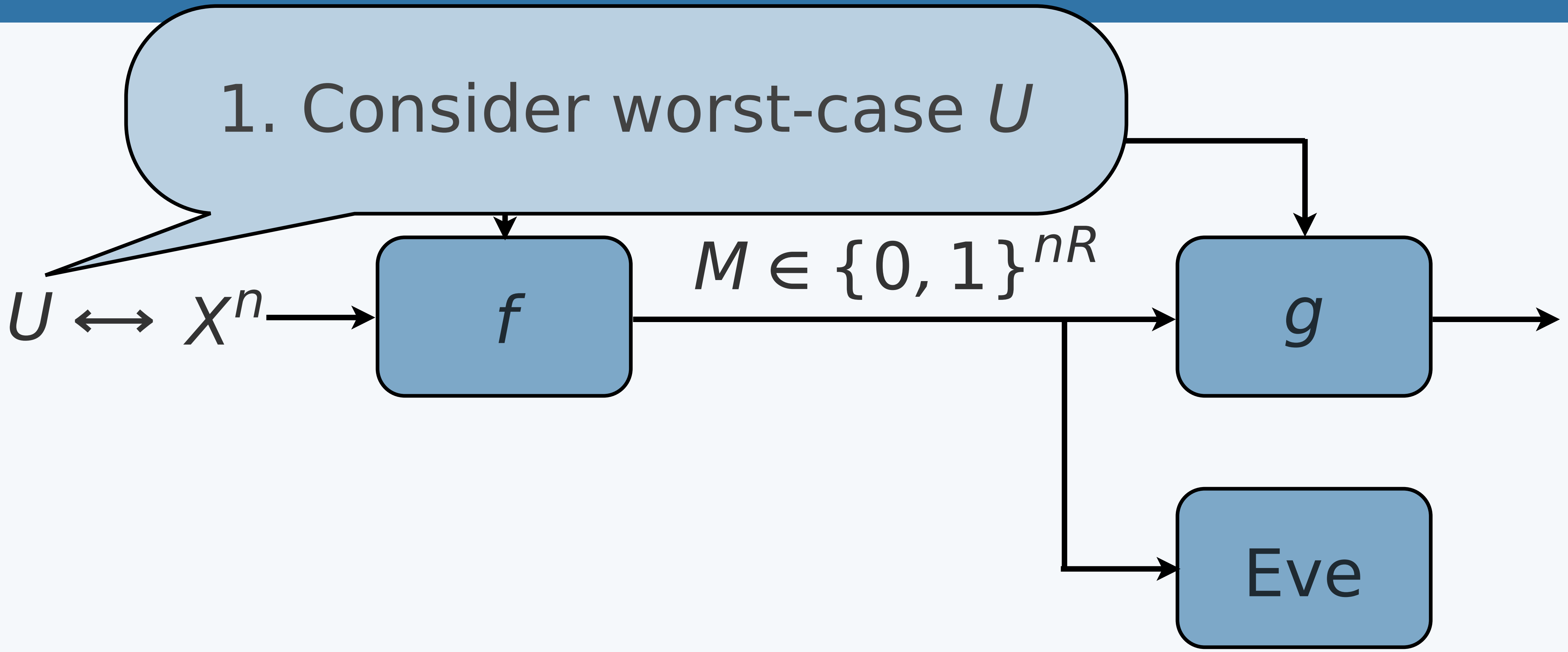
Achievability for Primary User



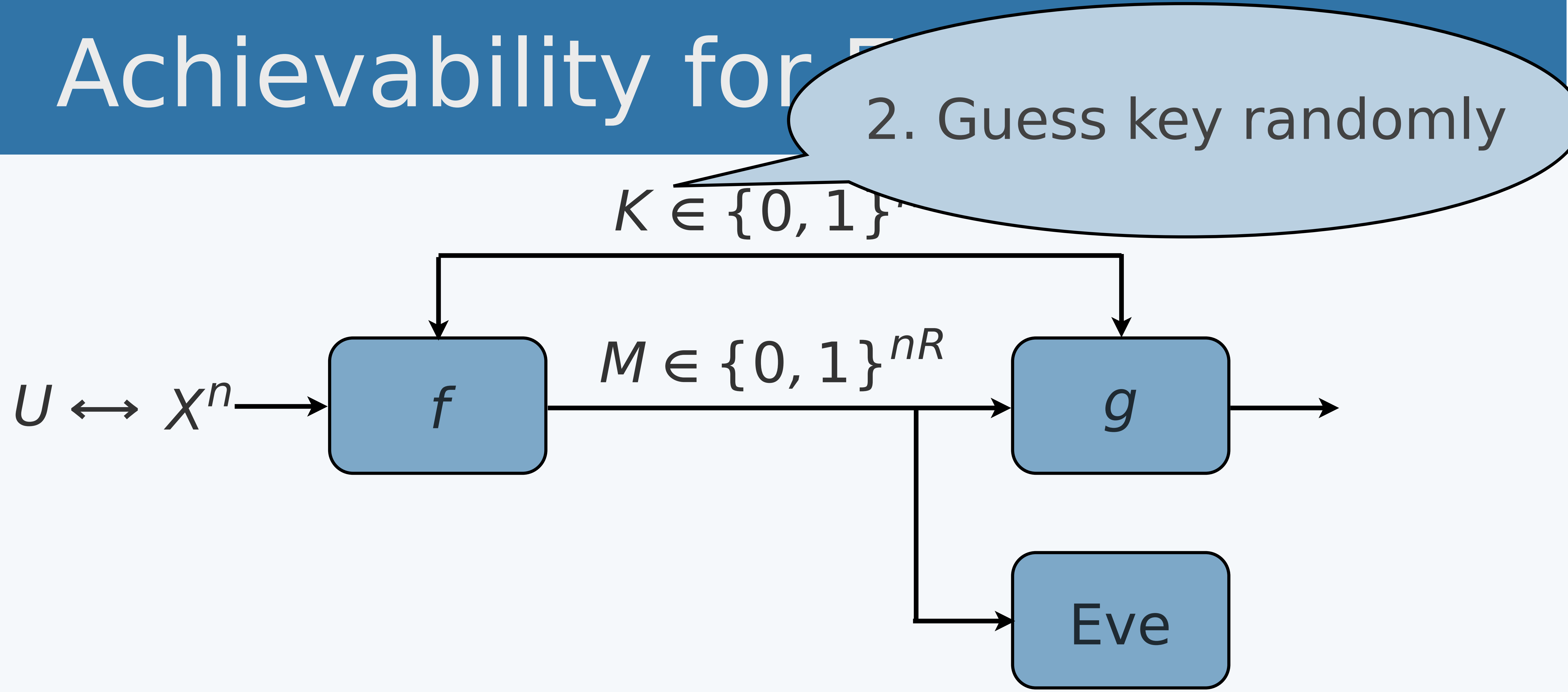
Achievability for Eavesdropper



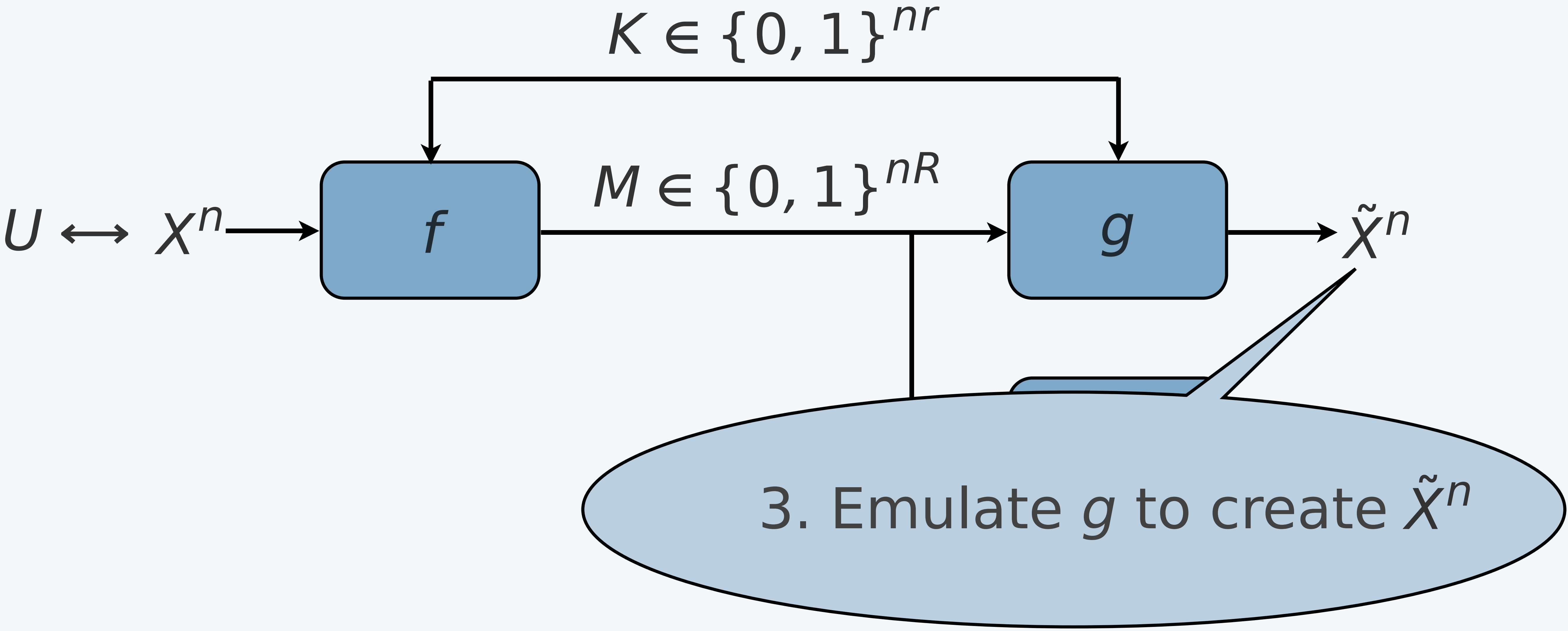
Achievability for Eavesdropper



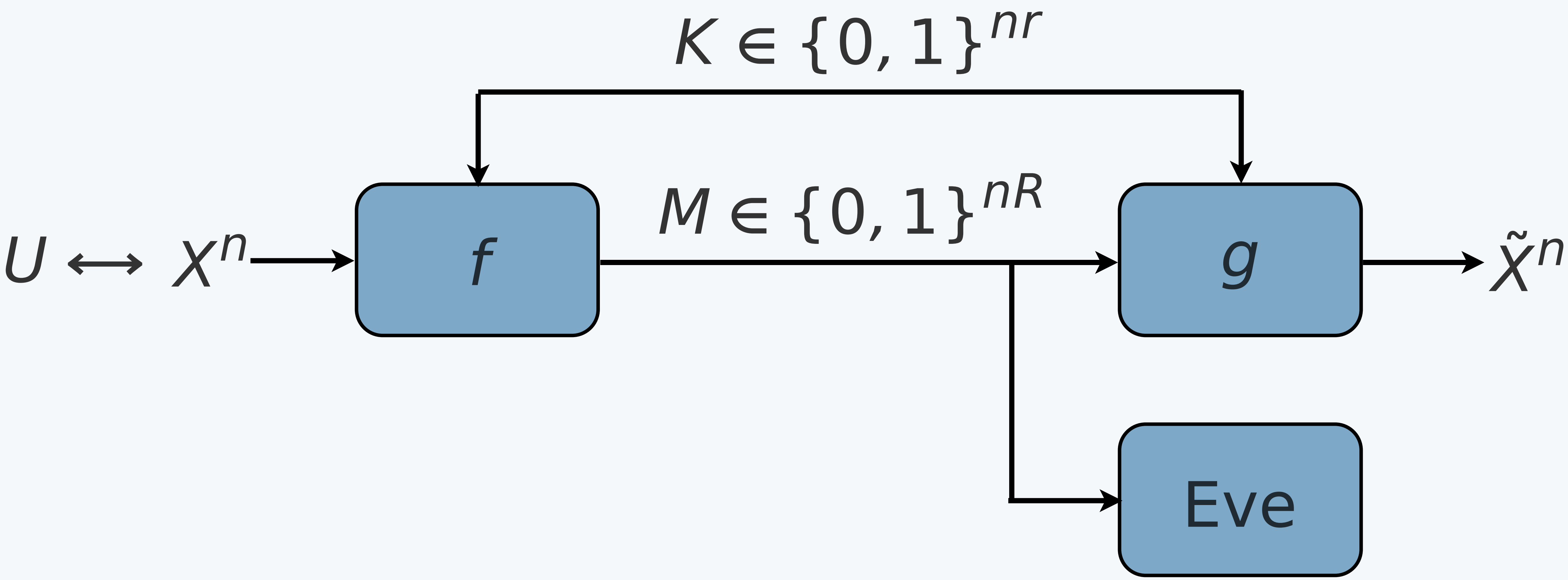
Achievability for E



Achievability for Eavesdropper

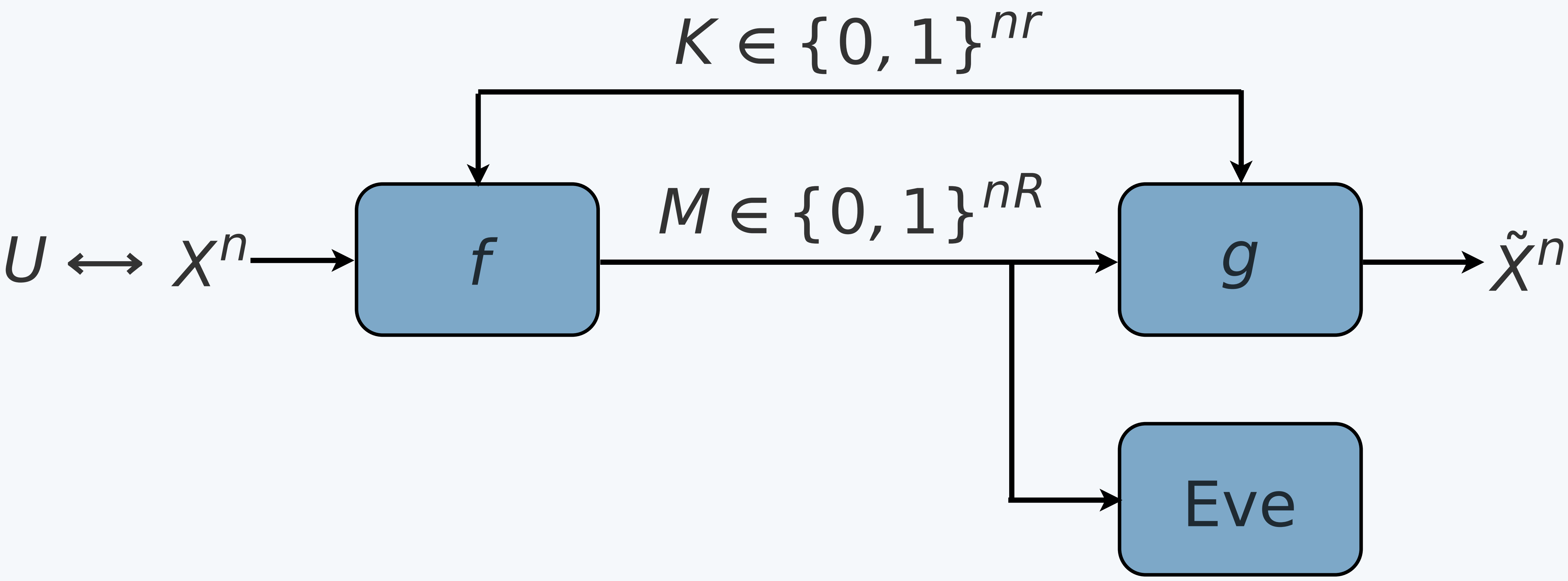


Achievability for Eavesdropper



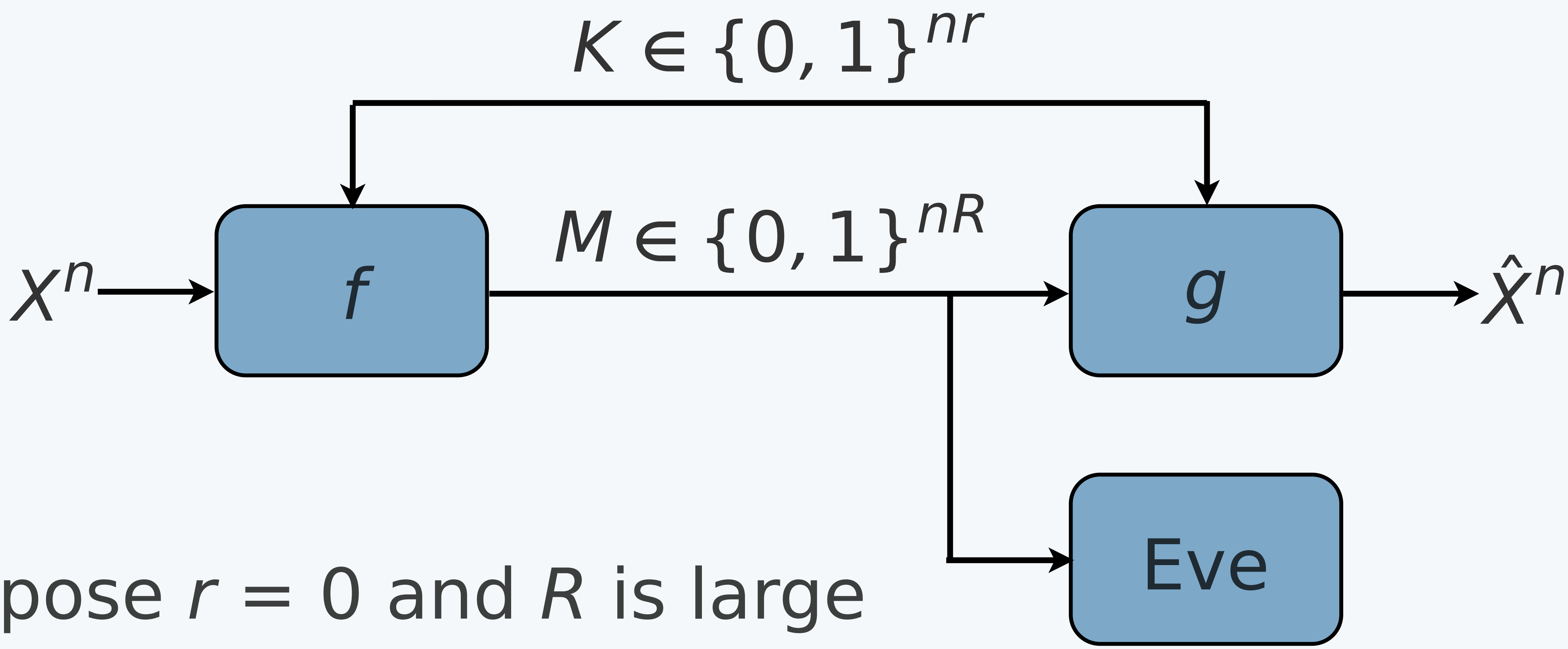
4. Pick X^n uniformly at random from within distortion ball around $\tilde{X}^n$.

Achievability for Eavesdropper



5. Generate U from X^n.

Quantization vs. Adding Noise



Suppose $r = 0$ and R is large

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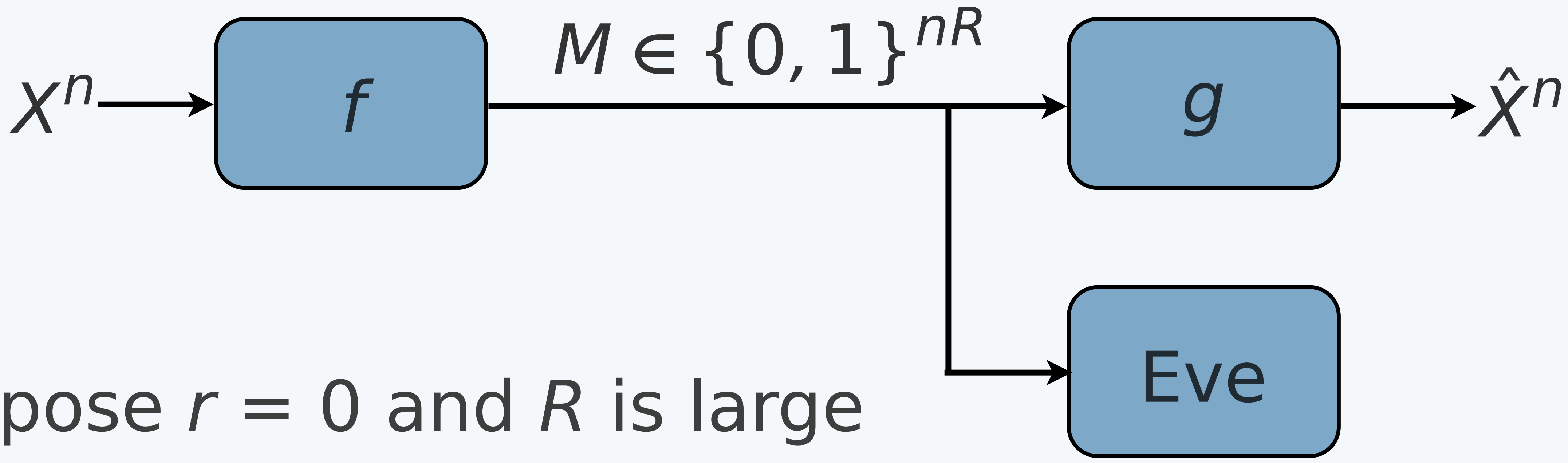

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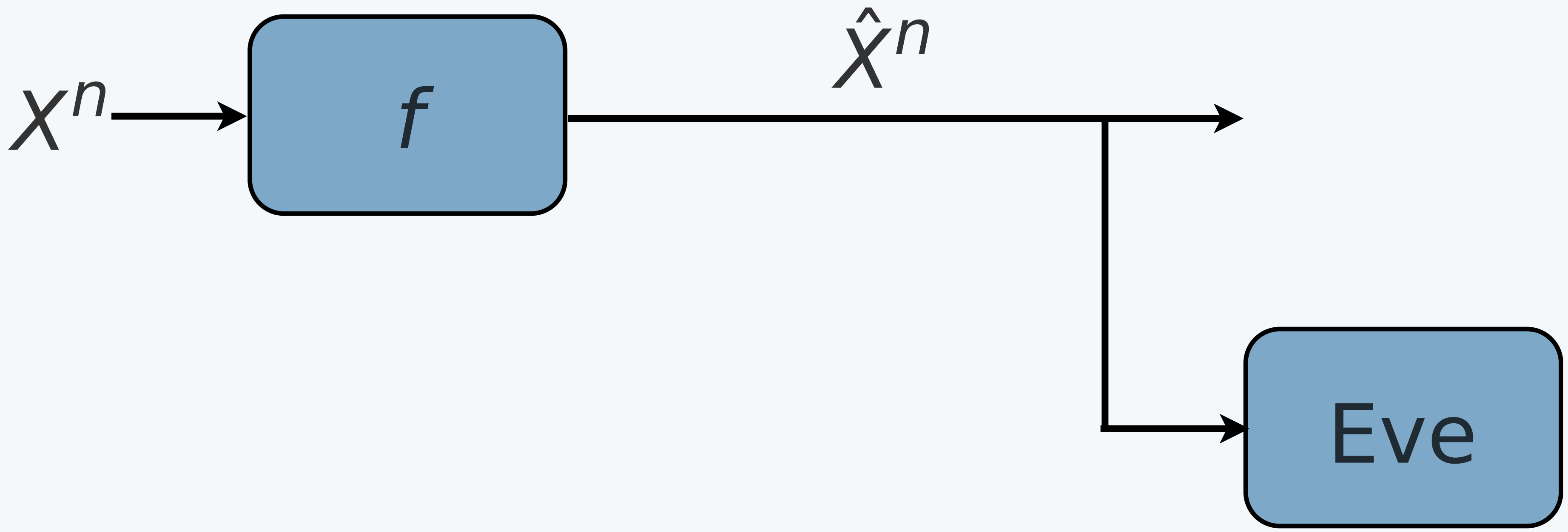

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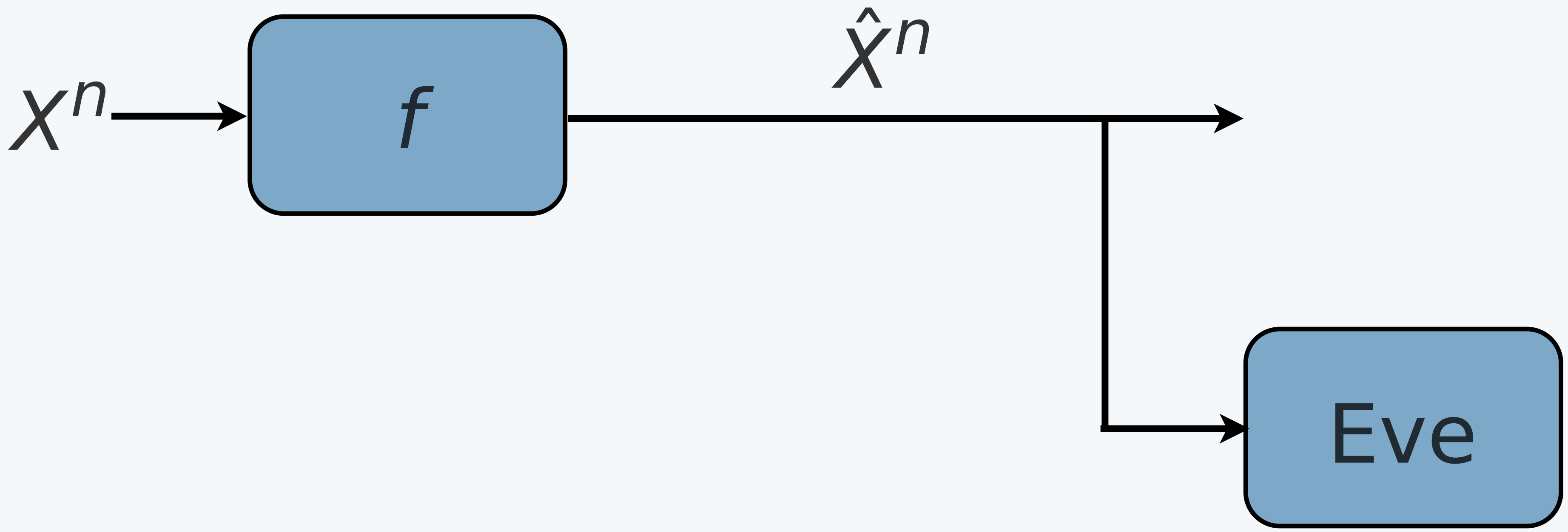

Then $\quad L_n = \min_{\hat{X}^n} \frac{1}{n} \cdot \mathcal{L}(X^n \to \hat{X}^n)$

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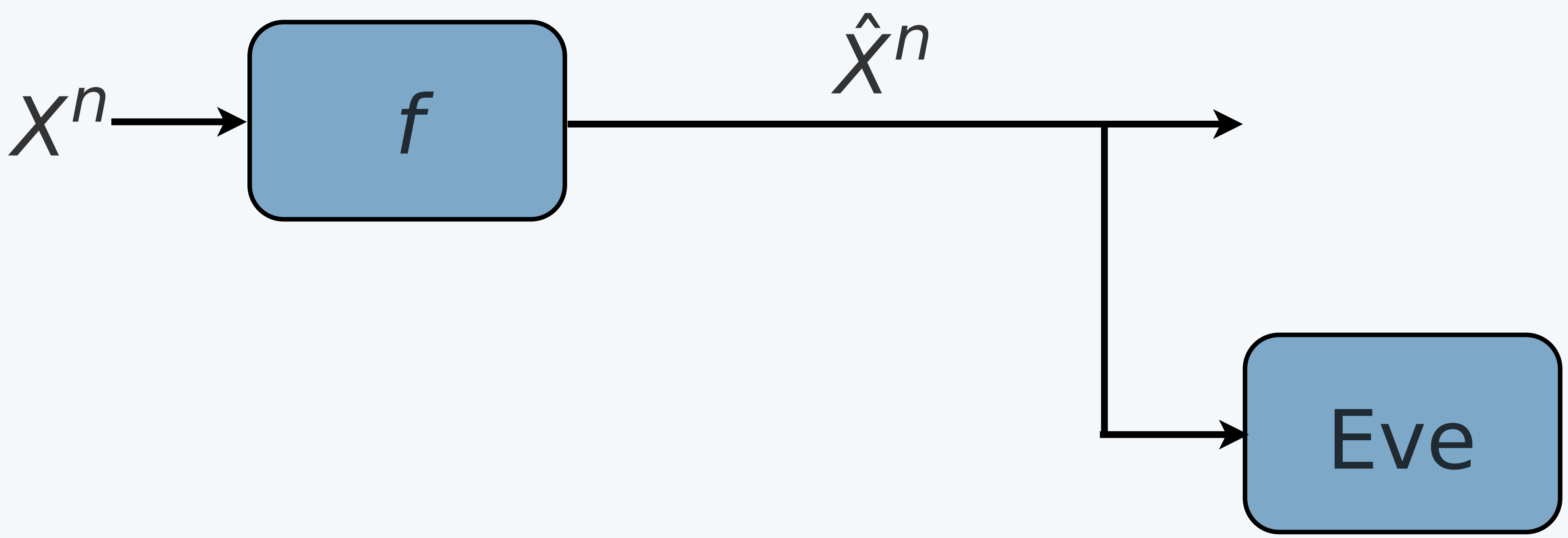

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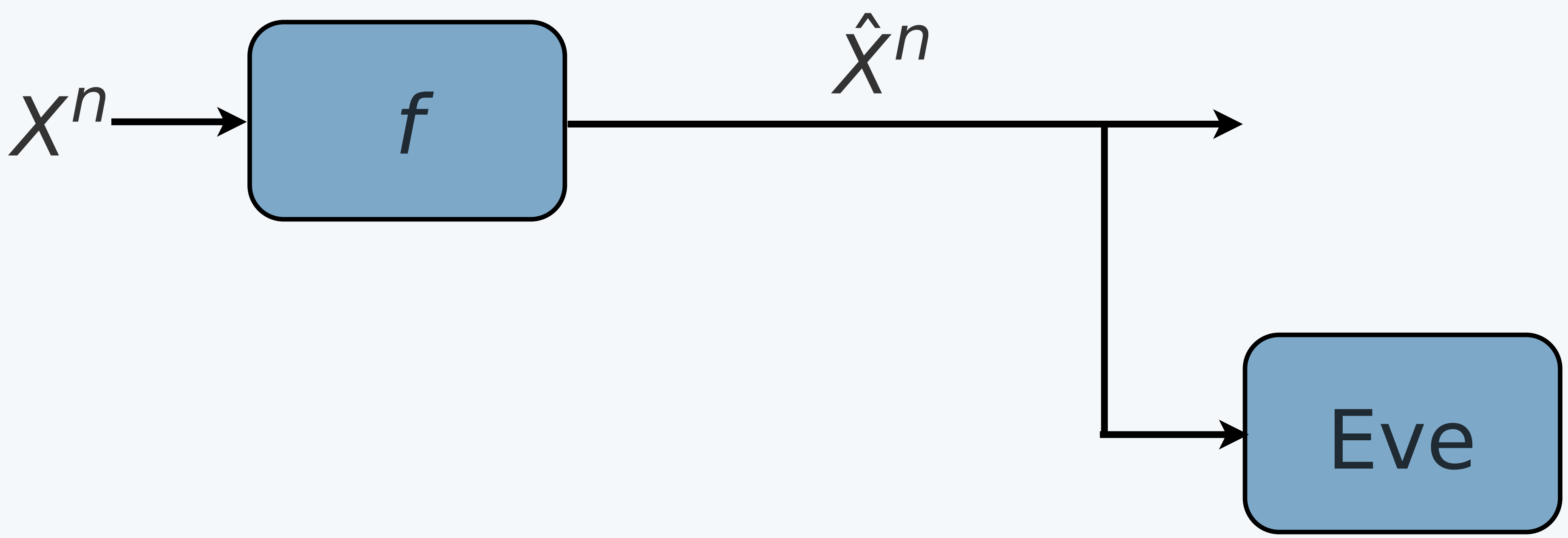

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[side channel]

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[cf. mutual info.]

Extension: Approx. Guessing

Def (Issa-Kamath-Wagner)**:** For any metric space $\mathcal{U}$,

$$\mathcal{L}_{\mathcal{U}}(X \to Y) = \sup_{\substack{U: U \leftrightarrow X \leftrightarrow Y \\ \exists u: \Pr(U \in B(u)) > 0}} \log \frac{\sup_{\hat{u}(.)} \Pr(U \in B(\hat{u}(Y)))}{\sup_{\hat{u}} \Pr(U \in B(\hat{u}))}$$

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Theorem (Issa-Kamath-Wagner)**:** For any metric space $\mathcal{U}$,

$$\mathcal{L}_{\mathcal{U}}(X \to Y) \leq \mathcal{L}(X \to Y)$$

with equality if $\mathcal{U}$ has countably many points no two of which are contained in the same unit ball.

Extension: General Gains

Def (Issa-Kamath-Wagner)**:**

$$\mathcal{L}_G(X \to Y) = \sup_{\substack{U : U \leftrightarrow X \leftrightarrow Y \\ g(\cdot,\cdot) : \mathcal{U} \times \hat{\mathcal{U}} \mapsto [0,\infty) : \\ \sup_{\hat{u}} E[g(U,\hat{u})] > 0}} \log \frac{\sup_{\hat{u}(\cdot)} E[g(U, \hat{u}(Y))]}{\sup_{\hat{u}} E[g(U, \hat{u})]}$$

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Theorem (Issa-Kamath-Wagner)**:** If X and Y are discrete, then

$$\mathcal{L}_G(X \to Y) = \mathcal{L}(X \to Y).$$

Opportunistic Attacks

Definition: The opportunistic maximal leakage is

$$\mathcal{L}_O(X \to Y) = \log E_Y \left[\sup_{U \leftrightarrow X \leftrightarrow Y} \frac{\sup_{\tilde{u}} P_{U|Y}(\tilde{u}|y)}{\sup_{\tilde{u}} P(\tilde{u})} \right]$$

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Theorem (Issa-Wagner): For any joint distribution P_{XY} on finite alphabets

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Extension: General Alphabet

Corollary (IKW)**:** If X and Y are jointly continuous then

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Corollary (IKW)**:** If X and Y are jointly Gaussian then

$$\mathcal{L}(X \to Y) = \begin{cases} 0 & \text{if } X, Y \text{ indep.} \\ \infty & \text{otherwise} \end{cases}$$

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