Theorem 12.17 Let K be an FCMI on $X_1, X_2, \cdots, X_n$. Then K holds if and only if $\mu^*(A) = 0$ for all $A \in Im(K)$.

Proof (due to Tie Liu)

A. "If"

Show that

$$H\left(X_{Q_1}, X_{Q_2}, \cdots, X_{Q_m} | X_T\right) = \sum_{i=1}^{m} H\left(X_{Q_i} | X_T\right).$$

B. "Only if"

1. An atom in $Im(K)$ has the form

$$\left(\bigcap_{i=1}^{k} \bigcap_{j \in W_i} \tilde{X}_j\right) - \tilde{X}_{T \cup (\cup_{i=1}^{k} (Q_i - W_i))}$$

where $W_i \subseteq Q_i$ and there exist at least two i such that $W_i \neq \emptyset$.

2. Consider the atom

$$A = \left(\bigcap_{i=1}^{m} \bigcap_{j \in Q_i} \tilde{X}_j\right) - \tilde{X}_T,$$

i.e., $W_i = Q_i$ for all i.

3. Then

$$\mu^*(A) = \sum_{J \subseteq \cup_i Q_i} (-1)^{|J|+1} H(X_J | X_T). \qquad (1)$$

4. For each $J \subseteq \cup_i Q_i$, let $J_i = J \cap Q_i$. Then

$$J = \bigcup_{i=1}^{m} J_i.$$

5. Since $X_{Q_1}, X_{Q_2}, \cdots, X_{Q_m}$ are mutually independent conditioning on X_T, so are $X_{J_1}, X_{J_2}, \cdots, X_{J_m}$.

6. Following (1), we have

$$\begin{aligned}
\mu^*(A) &= \sum_{J \subseteq \cup_i Q_i} (-1)^{|J|+1} \sum_{i=1}^{m} H\left(X_{J_i} | X_T\right) \\
&= \sum_{i=1}^{m} \sum_{J_i \subseteq Q_i} \alpha(J_i) H\left(X_{J_i} | X_T\right),
\end{aligned}$$

where $\alpha(J_i)$ is the coefficient of $H\left(X_{J_i} | X_T\right)$ in the summation.

7. By simple counting, we obtain that

$$\begin{aligned}
\alpha(J_i) &= \sum_{L \subseteq \cup_{j \neq i} Q_j} (-1)^{|J_i|+|L|+1} \\
&= (-1)^{|J_i|+1} \sum_{L \subseteq \cup_{j \neq i} Q_j} (-1)^{|L|} \\
&= 0,
\end{aligned}$$

because

$$\sum_{L \subseteq \cup_{j \neq i} Q_j} (-1)^{|L|} = 0$$

by the binomial formula. Hence $\mu^*(A) = 0$.