Theorem 12.25 $X_1, X_2, \cdots, X_n$ form a Markov graph G if and only if μ^* vanishes on all the Type II atoms.

Proof

1. Note that $\{U_A, A \in \mathcal{A}\}$ contains precisely all the proper subsets of $\mathcal{N}_n$.

2. Thus the set of FCMI's specified by the graph G can be written as

$$[U_A : A \in T_2]$$

3. By Theorem 12.19, it suffices to prove that

$$Im([U_A : A \in T_2]) = T_2.$$

A. "$T_2 \subseteq Im([U_A : A \in T_2])$"

1. Consider an atom $A \in T_2$.

2. An atom in $Im([U_A])$ has the form

$$\left(\bigcap_{i=1}^{s(U_A)} \bigcap_{j \in W_i} \tilde{X}_j \right) - \tilde{X}_{U_A \cup \left(\bigcup_{i=1}^{s(U_A)} (V_i(U_A) - W_i) \right)}$$

where $W_i \subseteq V_i(U_A)$ and $W_i \neq \emptyset$ for at least two i.

3. Letting $W_i = V_i(U_A)$ for all $1 \leq i \leq s(U_A)$, we see that

$$A \in Im([U_A]) \subseteq Im([U_{A'} : A' \in T_2]).$$

B. "$Im([U_A : A \in T_2]) \subseteq T_2$"

1. Consider $A \in Im([U_{A'} : A' \in T_2])$. Then there exists $A^* \in T_2$ such that $A \in Im([U_{A^*}])$.

2. Following Step 2 in **Part A**, A is equal to

$$\left(\bigcap_{i=1}^{s(U_{A^*})} \bigcap_{j \in W_i} \tilde{X}_j \right) - \tilde{X}_{U_{A^*} \cup \left(\bigcup_{i=1}^{s(U_{A^*})} (V_i(U_{A^*}) - W_i) \right)}$$

where $W_i \subseteq V_i(U_{A^*})$ and $W_i \neq \emptyset$ for at least two i.

3. Then

$$U_A = U_{A^*} \cup \bigcup_{i=1}^{s(U_{A^*})} (V_i(U_{A^*}) - W_i).$$

4. By Proposition 12.26, we see that those (at least two) W_i which are nonempty are disjoint in $G \backslash U_A$. This implies $s(U_A) > 1$, i.e., $A \in T_2$.