Information Rates for Phase Noise Channels

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Joint collaboration with:

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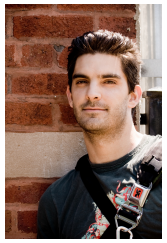
- Stefano Rini (National Chiao Tung University)



Outline

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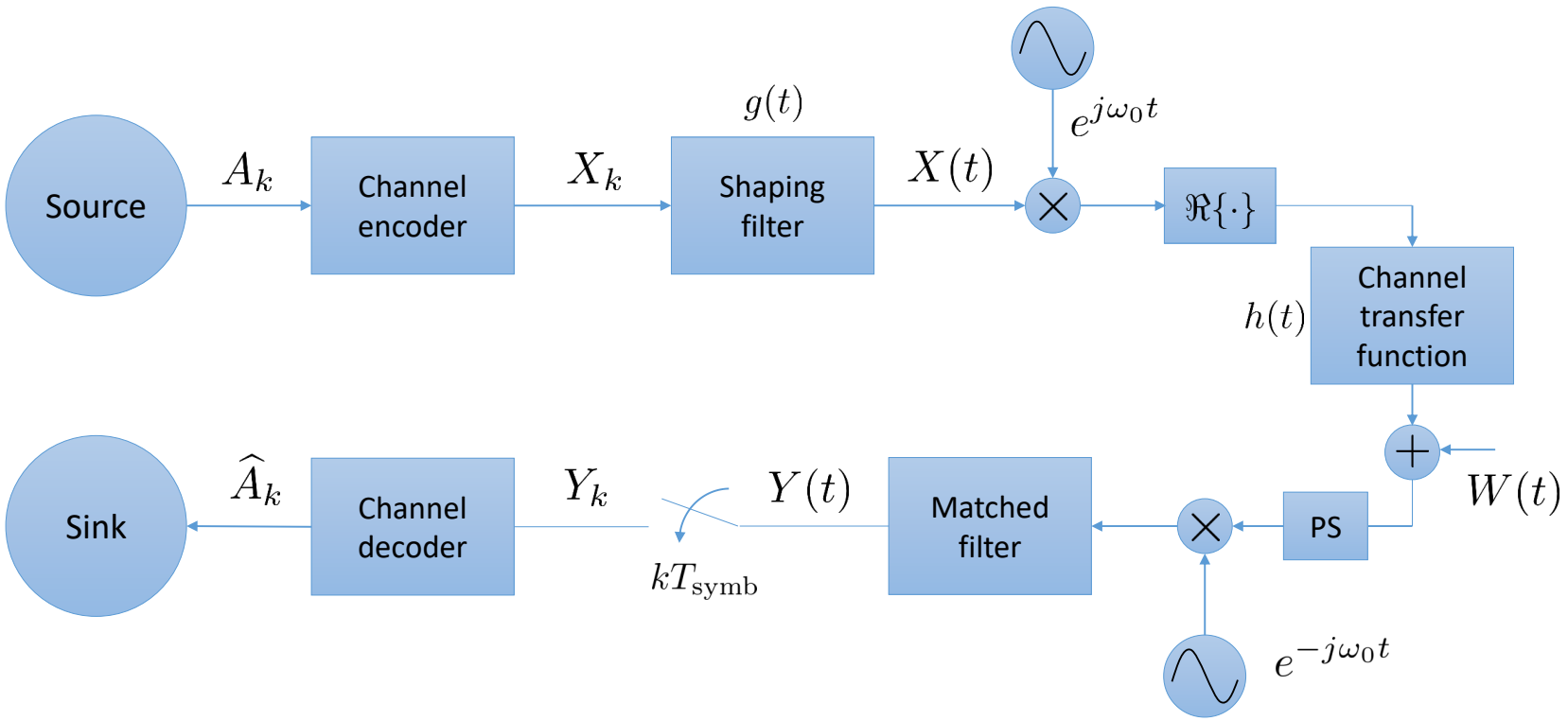
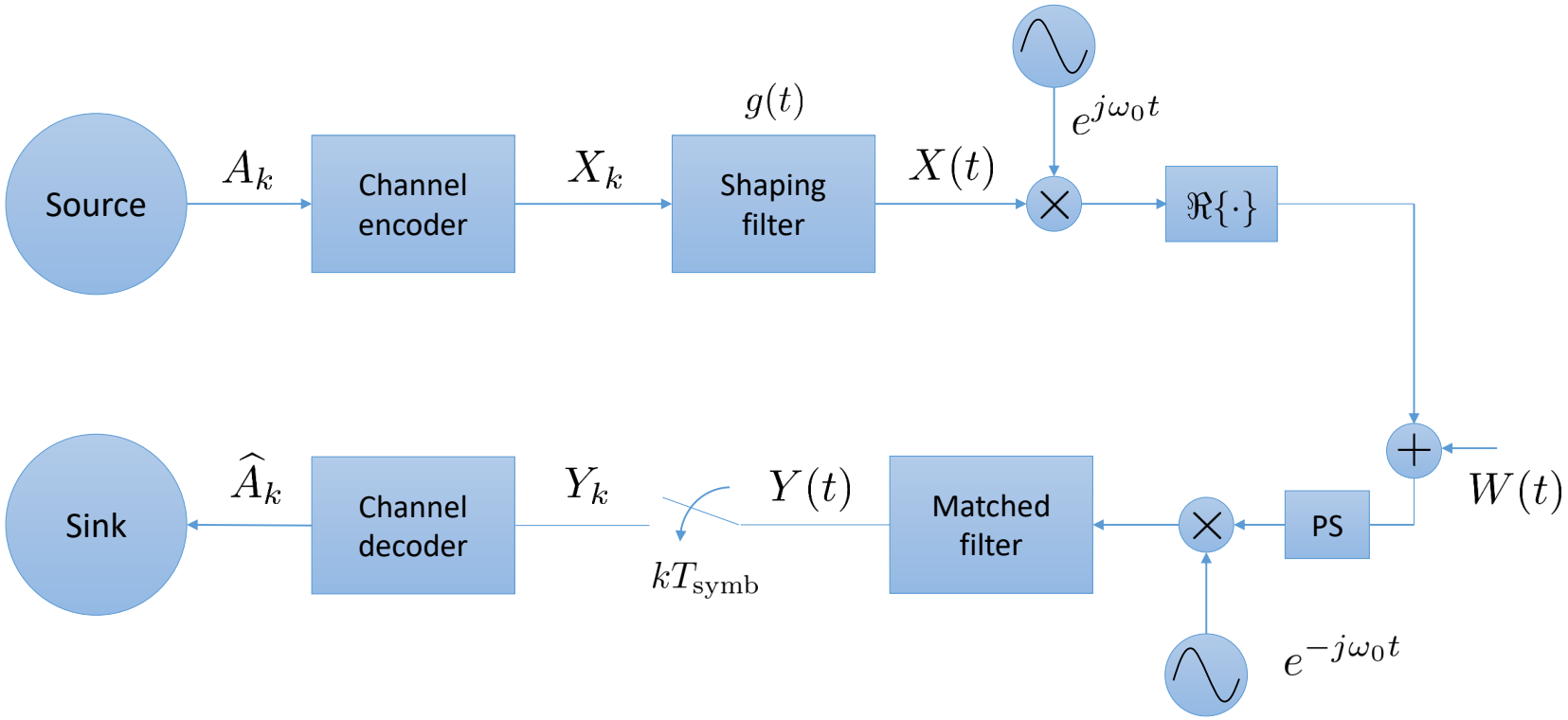
A Classic Communication Scheme



The AWGN Channel



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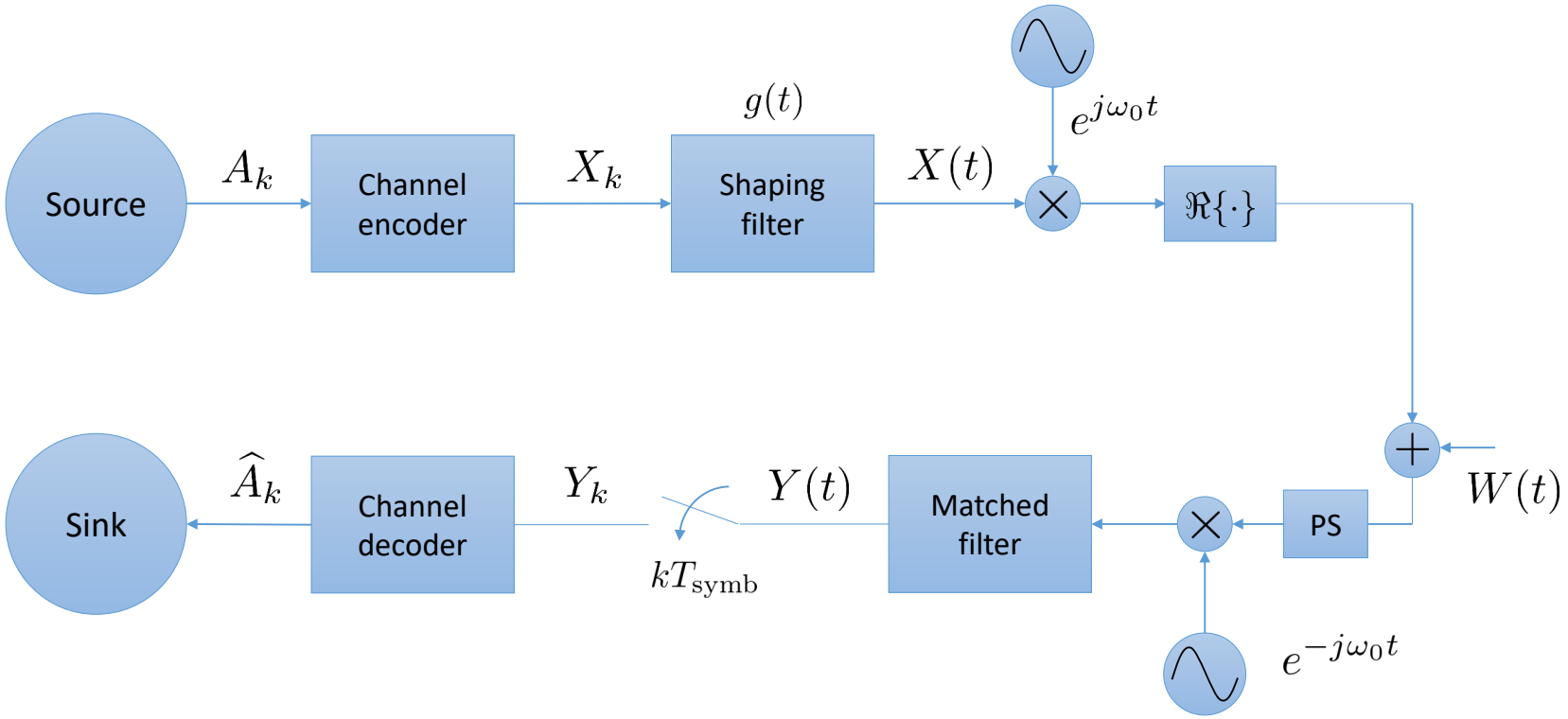
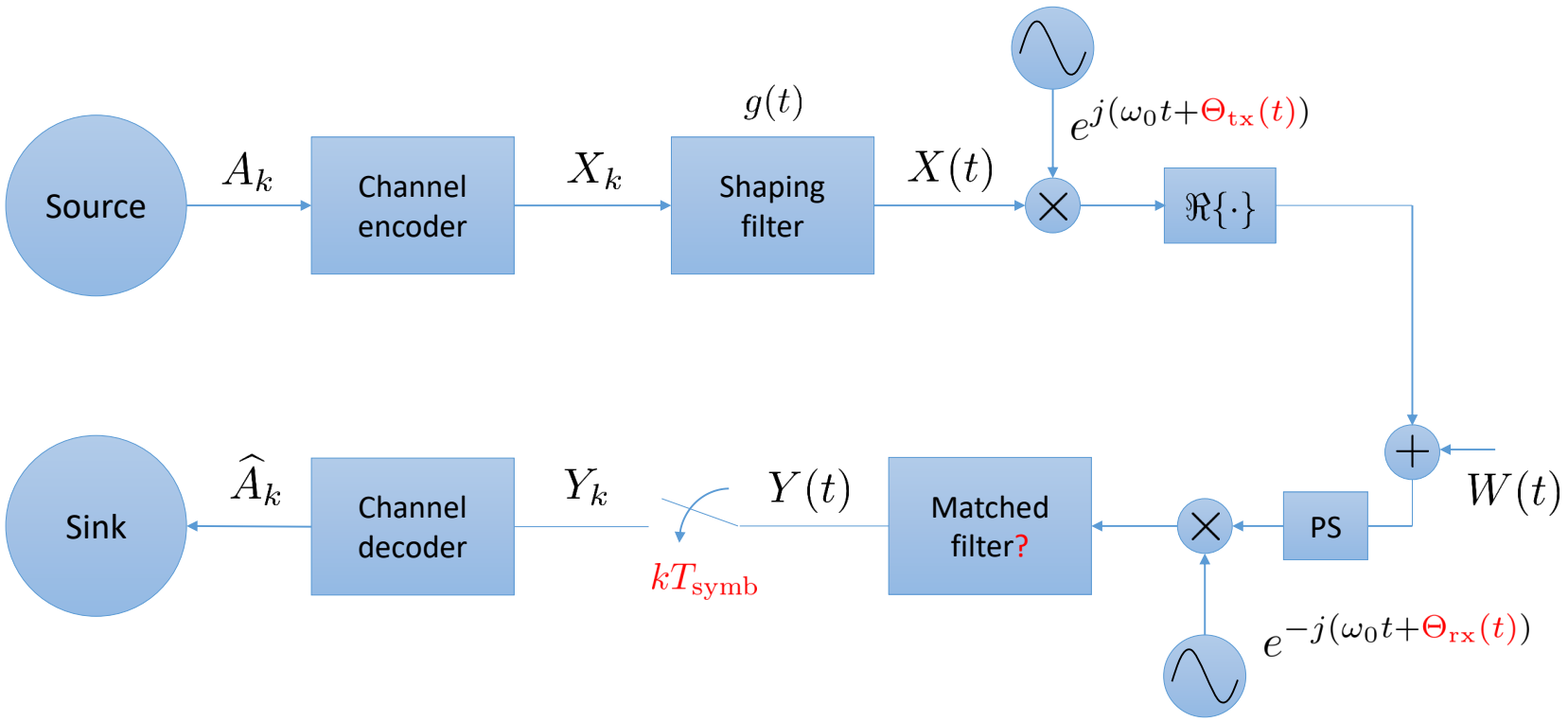


$$Y_k = X_k + W_k$$

The Actual AWGN Channel



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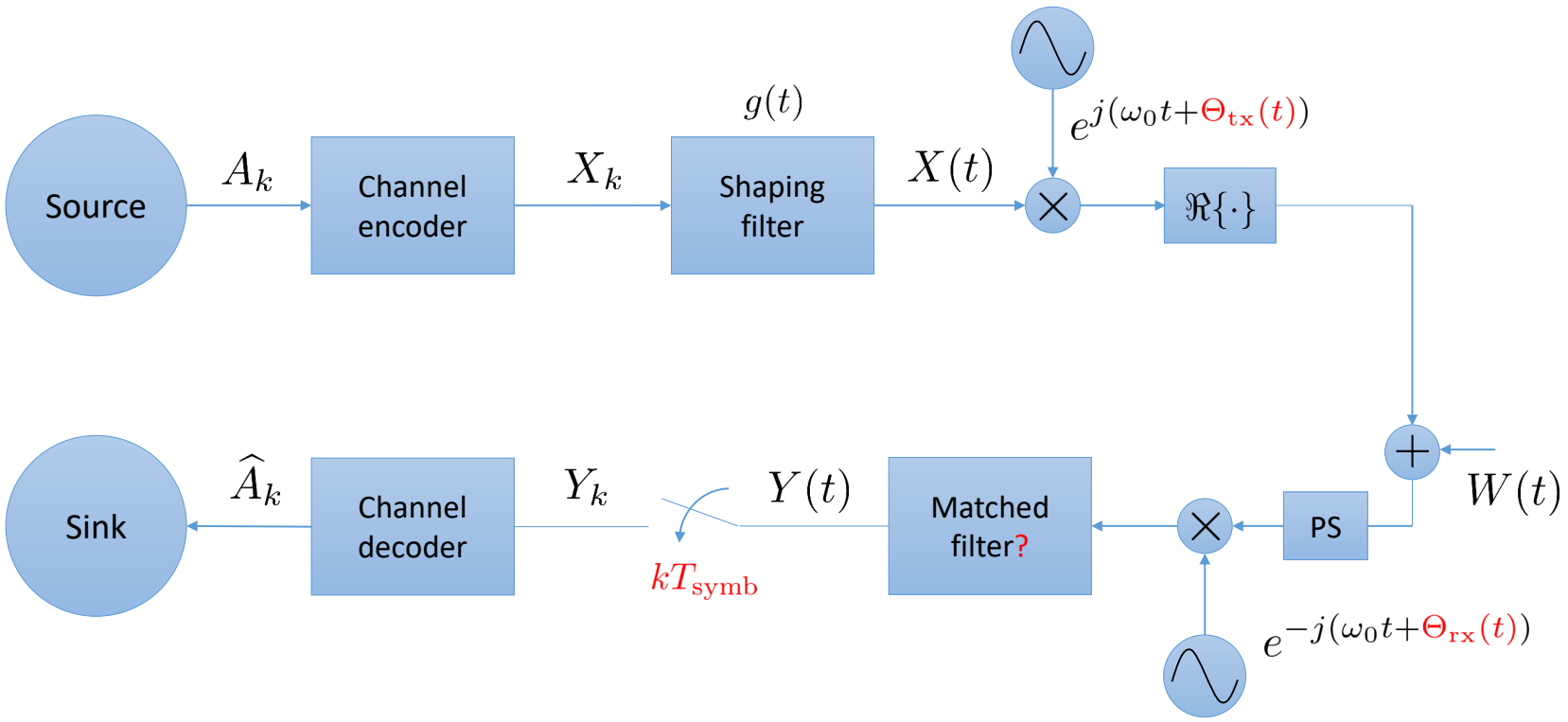

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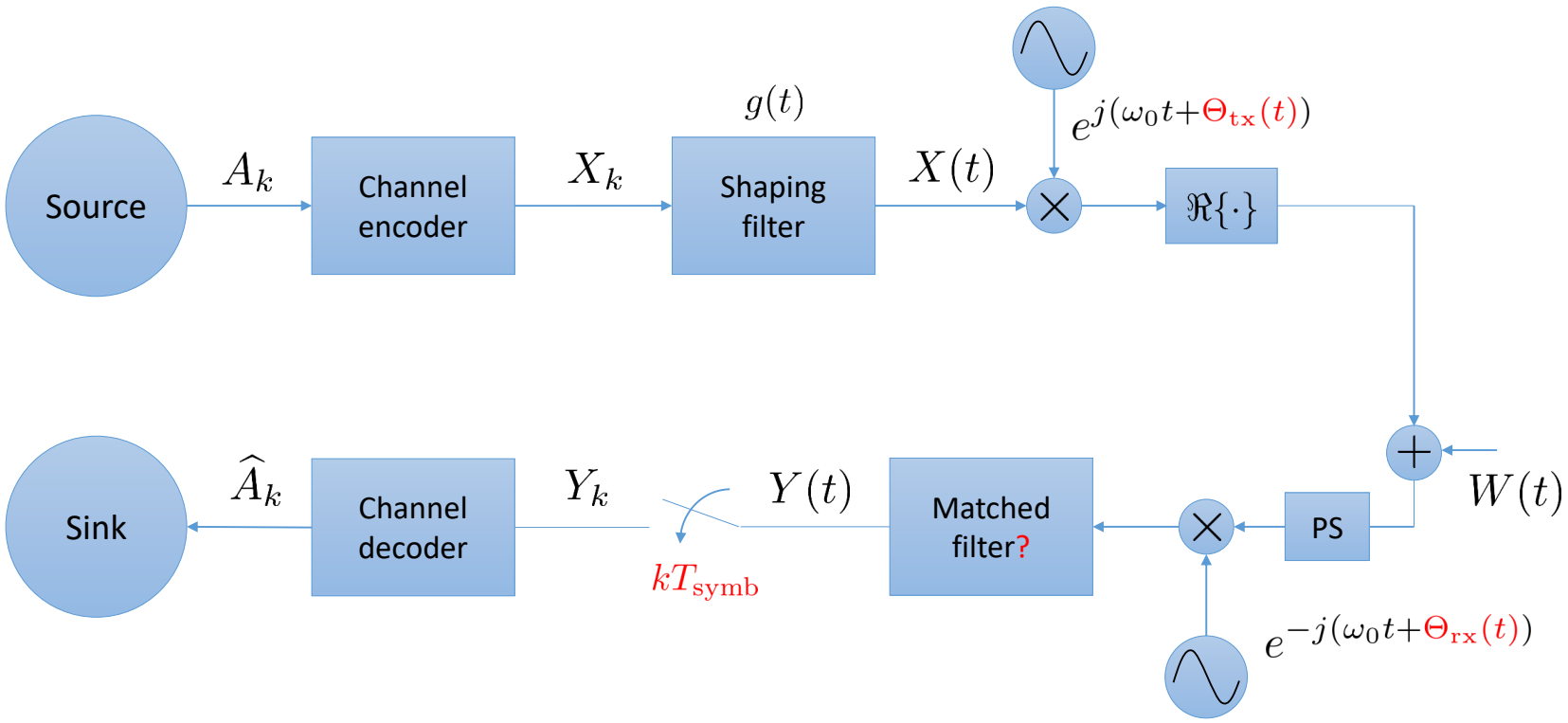

Phase noise processes Θ_{tx} and Θ_{rx}
Sampling at kT_{symb} is no longer optimal!

Phase Noise Channels

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- Fast and slow fading effects in wireless environments
- Nonlinear propagation effects in fiber-optic commun. (amplitude mod. is converted into phase mod. $\longrightarrow$ phase-noise strength depends also on signal amplitude)

Phase Noise Channels

Questions:

- Models for phase noise channels

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- Signal and code design

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Representation of continuous-time waveforms

- Assume input waveform $\{X(t)\}_{t=0}^{T}$ is square integrable:

$$X(t) \in \mathcal{L}^2[0, T] \longrightarrow \int_0^T |X(t)|^2 dt < \infty$$

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- Equivalent representations: $\{X(t)\}_{t=0}^{T} \Longleftrightarrow X_1 X_2 \cdots$

Mutual information for random waveforms

- The average mutual information between $\{X(t)\}_{t=0}^{T}$ and $\{Y(t)\}_{t=0}^{T}$ is [Gallager, 1968]

$$I\left(\{X(t)\}_{t=0}^{T};\{Y(t)\}_{t=0}^{T}\right)=\lim_{n\to\infty} I\left(X_1\cdots X_n;Y_1\cdots Y_n\right)$$

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$$Y(t) = X(t)e^{j\Theta(t)} + W(t)$$

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$$\begin{aligned} Y_m &= \int_0^T Y(t)\phi_m(t)^\star dt = \int_{(m-1)\Delta}^{m\Delta} \frac{Y(t)}{\sqrt{\Delta}} dt \\ &= X_m \int_{(m-1)\Delta}^{m\Delta} \frac{e^{j\Theta(t)}}{\Delta} dt + W_m \end{aligned}$$

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- How different are the two models in terms of capacity?

Outline

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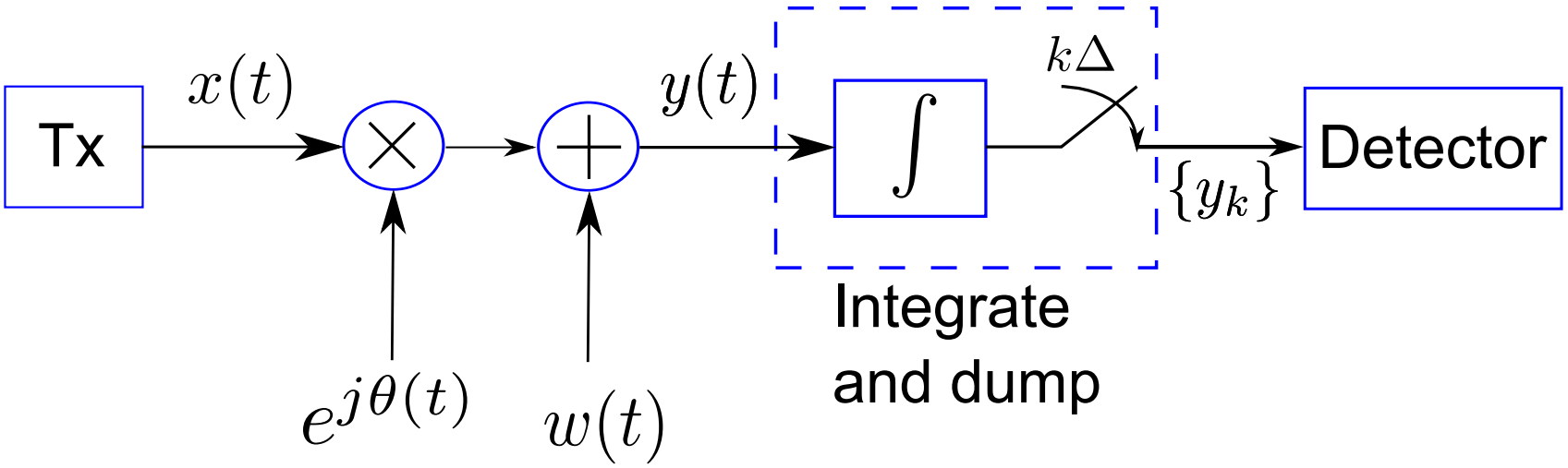

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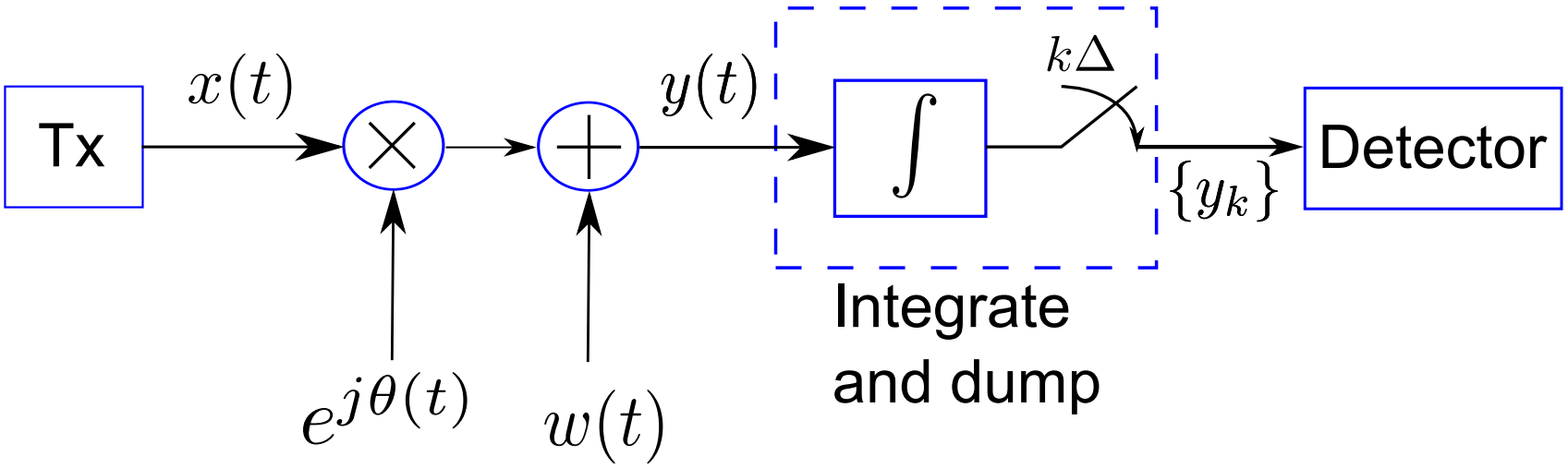

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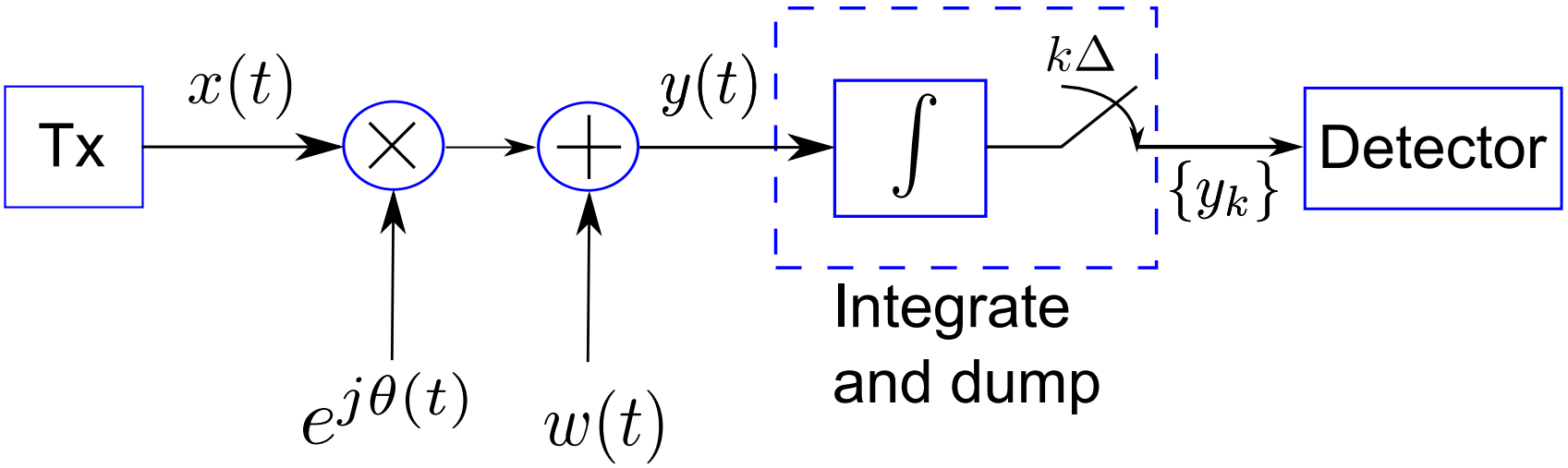

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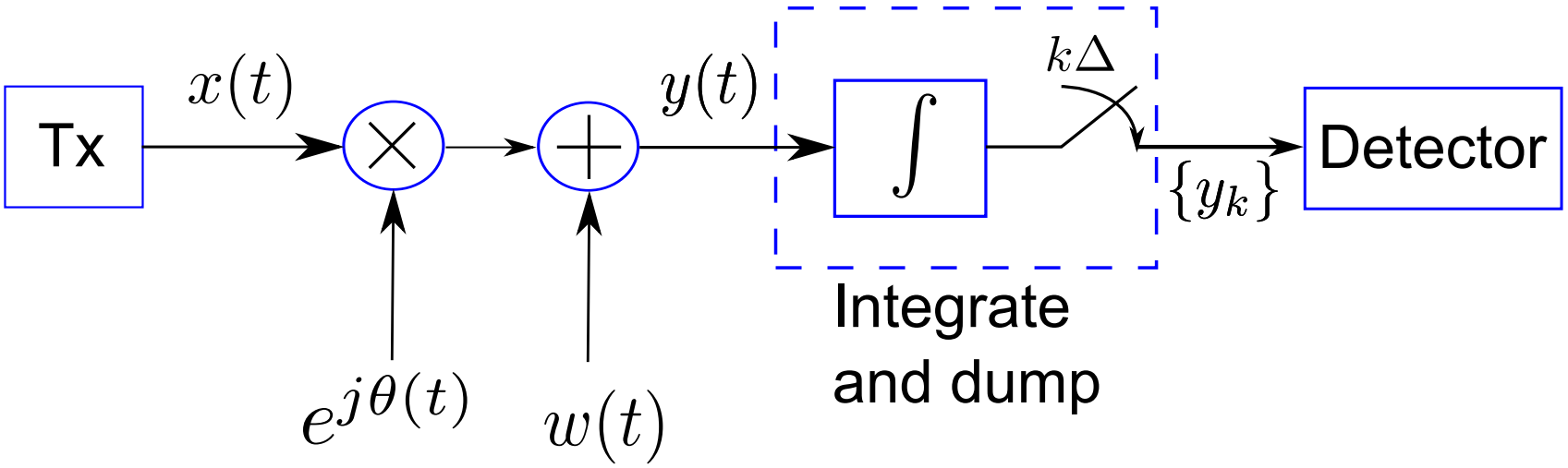

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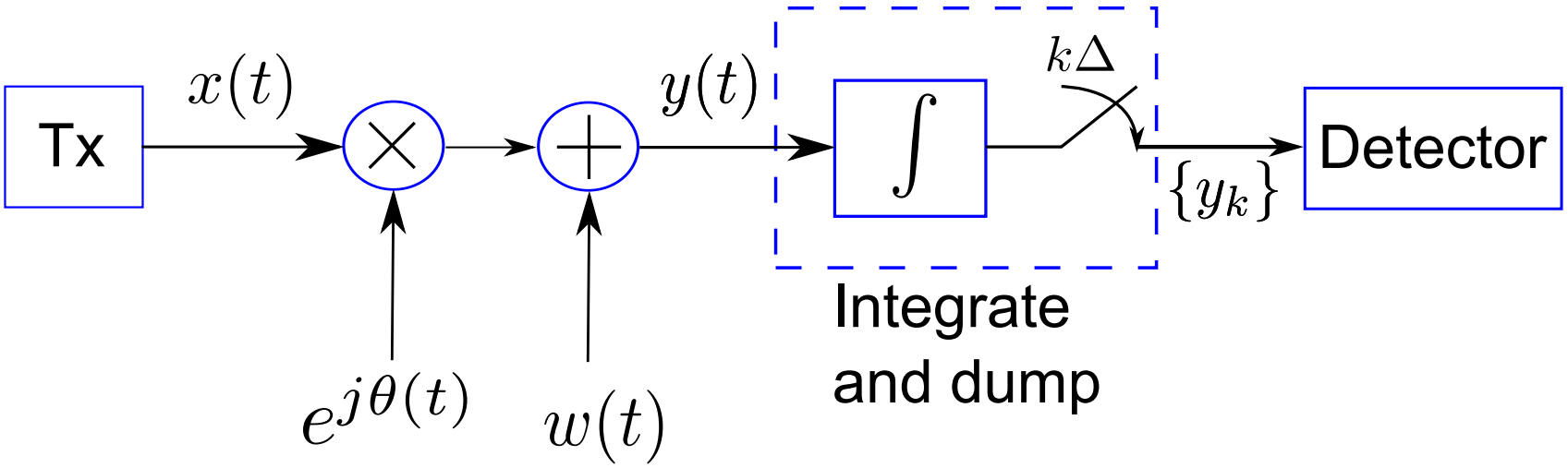

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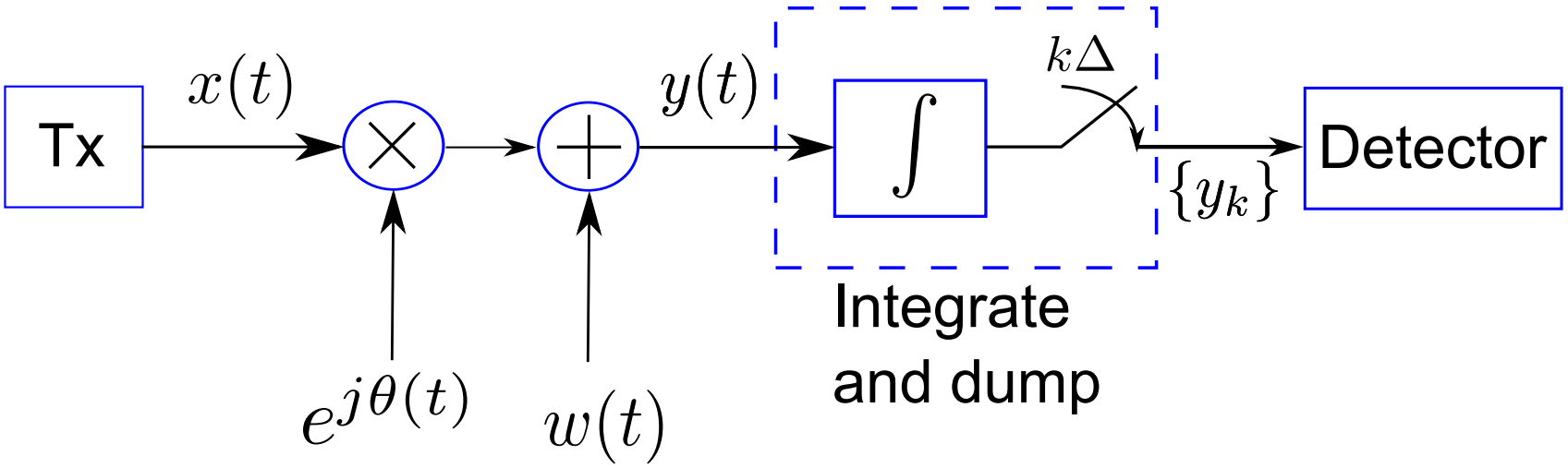

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Oversampled channel model

$$Y_{mL+\ell} = X_m \, F_{mL+\ell} + W_m, \quad \ell = 1, \ldots, L$$
$$m = 1, \ldots, n$$

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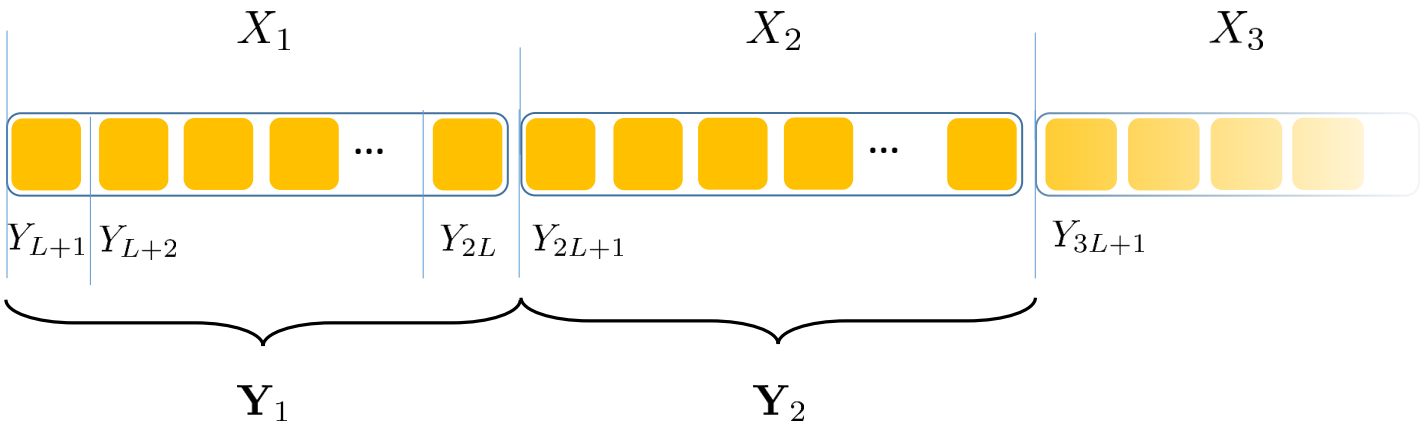

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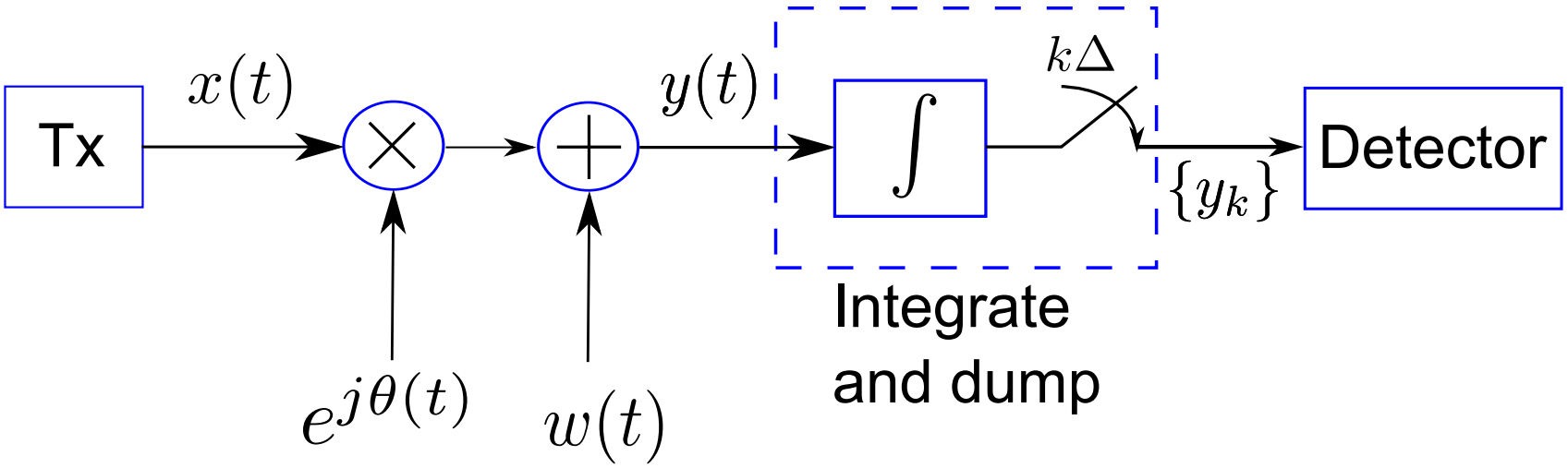

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- Widely used for modeling oscillators
- Also known as Brownian motion:

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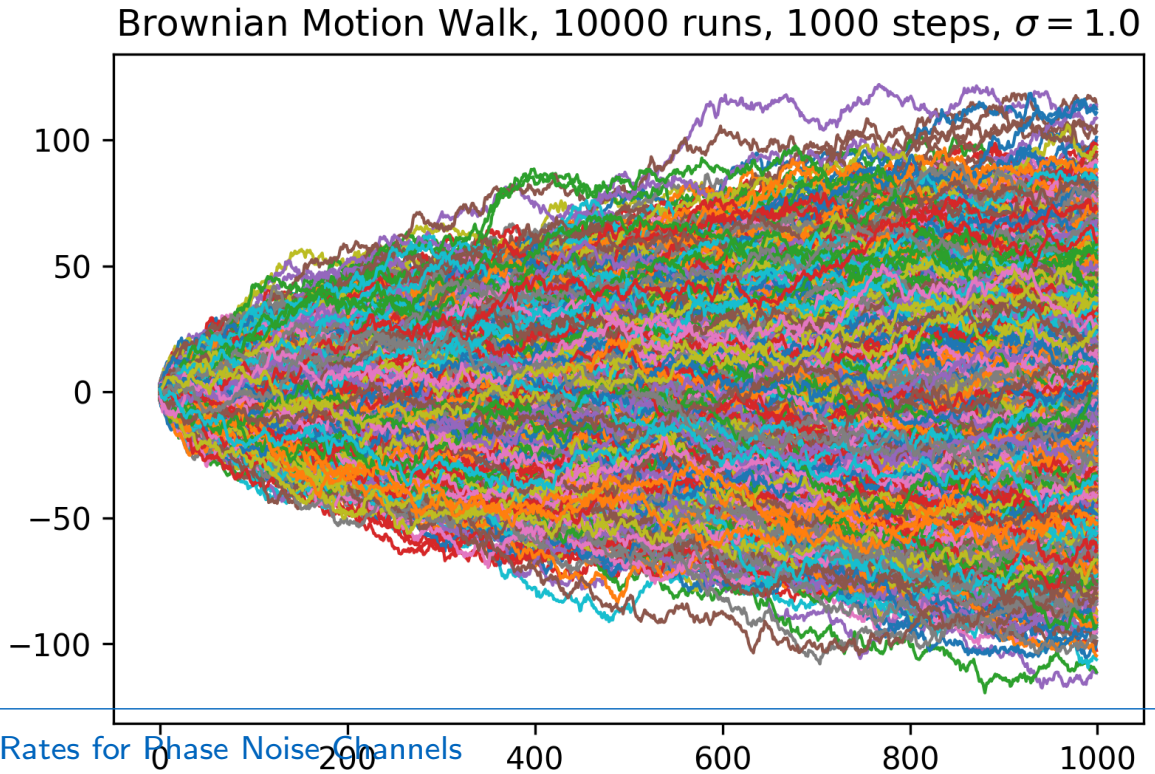

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- Oversampling, i.e. $L > 1$, increases information rates

Wiener phase noise channel

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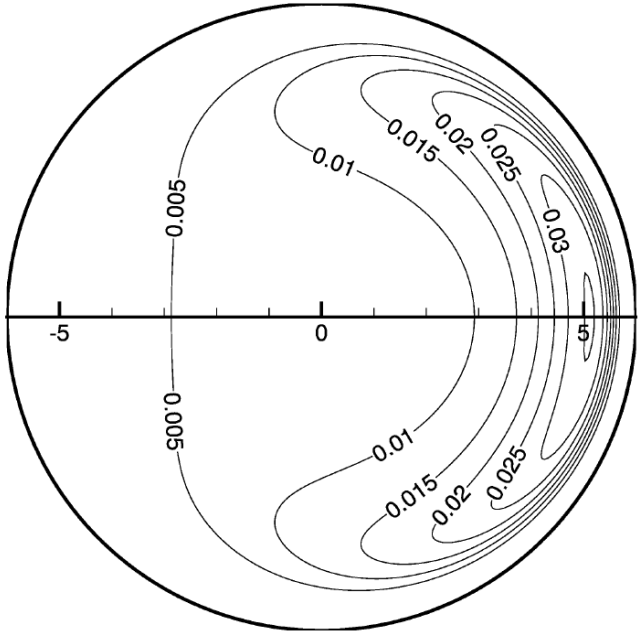

Contour plot of the unnormalized fading pdf for $\Delta = 6$ and $\gamma = 1$. (Y. Wang *et al.*, TCOM 2006, vol. 54, no. 5)

Oversampled Discrete-time Wiener Phase Noise Channel

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$$\mathcal{C}(P,\Delta,\gamma) = \lim_{n\to\infty} \sup_{F_{X_m}:\, \mathrm{E}[|X_m|^2]\leq P\Delta} \frac{1}{n} I\left(X_1 \cdots X_n\,; \mathbf{Y}_1 \cdots \mathbf{Y}_n\right)$$

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- We assume iid X_m's with $\underline{/X_m} \sim \mathcal{U}[0, 2\pi)$

A capacity upper bound

Define $X_1^n = X_1 \cdots X_n$

$$\frac{1}{n} I\left(X_1^n ; \mathbf{Y}_1^n\right) = \frac{1}{n} \sum_{m=1}^{n} I\left(X_1^n ; \mathbf{Y}_m \,|\, \mathbf{Y}_1^{m-1}\right)$$

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\text{(Markov)} &= \frac{1}{n} \sum_{m=1}^{n} I\left(X_m; \mathbf{Y}_m \,|\, \Theta_{mL+1}\right) + I\left(\Theta_{mL+1}; \mathbf{Y}_m \,|\, \mathbf{Y}_1^{m-1}\right)
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\end{aligned}$$

A capacity upper bound

Define $X_1^n = X_1 \cdots X_n$

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Polar decomposition

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  - Application of the I-MMSE formula to the phase mod. term

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\mathcal{C} \leq \frac{1}{2}\log\left(1+\frac{P}{2}\right) + \log(2\pi) + \frac{1}{2}\log\left(P\Delta\left(\sqrt{1+\frac{4}{\gamma^2 P\Delta^2}}-1\right)\right)
$$

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Define the degrees of freedom as

$$D(\alpha) = \lim_{P \to \infty} \frac{\mathcal{C}(P, \Delta = P^{-\alpha}, \gamma)}{\log(P)}$$

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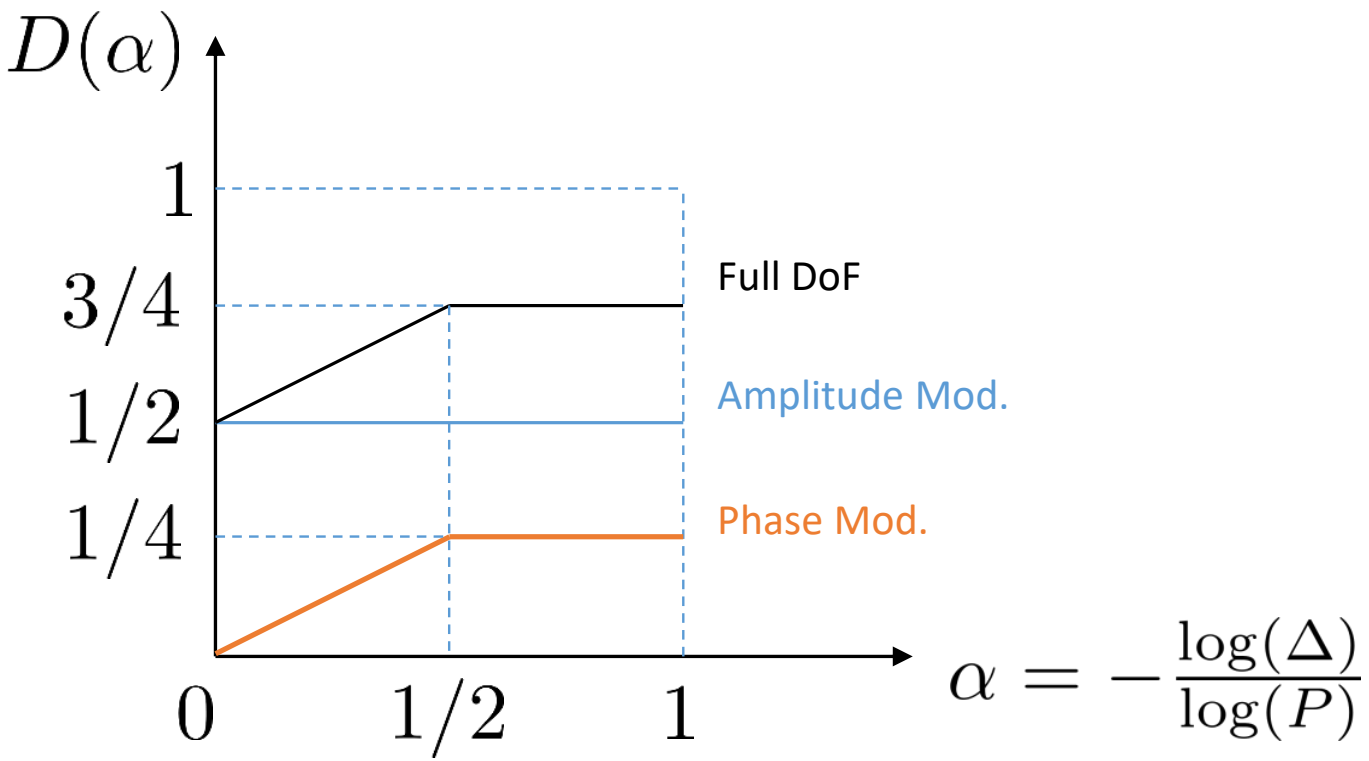

Outline

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  - The fundamental tension between additive noise and phase noise limits the degrees of freedom

Thanks for your attention!

Derivation of the average power constraint

$$
\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\int_0^T |X(t)|^2 \,\mathrm{d}t &= \mathsf{E}\left[\frac{1}{T}\int_0^T |X(t)|^2 \,\mathrm{d}t\right] \\
&= \mathsf{E}\left[\frac{1}{T}\int_0^T \left|\sum_{m=1}^{n} X_m \sum_{\ell=1}^{L} \phi_{mL+\ell}(t)\right|^2 \mathrm{d}t\right] \\
\text{(orthogonality)} &= \mathsf{E}\left[\frac{1}{T}\sum_{m=1}^{n}\sum_{\ell=1}^{L}\int_{(mL+\ell-1)\Delta}^{(mL+\ell)\Delta} \frac{|X_m|^2}{\Delta}\,\mathrm{d}t\right] \\
&= \frac{Ln}{T}\mathsf{E}\left[|X_m|^2\right] \\
&= \frac{1}{\Delta}\mathsf{E}\left[|X_m|^2\right] \leq P
\end{aligned}
$$