Lattice Index Coding

Part I - How to Utilize Side Information at the PHY Layer

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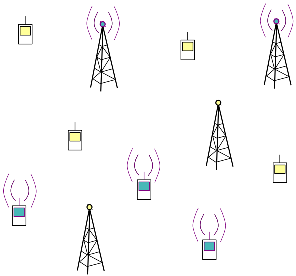
Communication in Wireless Networks



Defining features

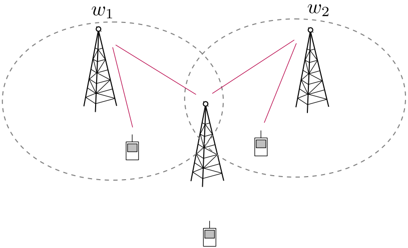
1. *Interference*:
   Signals from multiple transmitters superimpose.

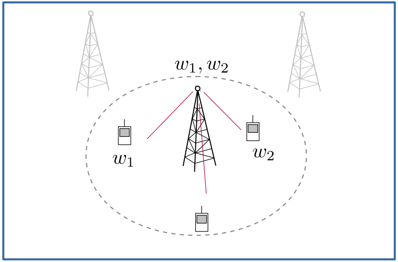
2. *Broadcast*:
   A transmit signal is heard by multiple receivers, with different signal strengths.

- Methods to handle interference:
  Han-Kobayashi coding, interference alignment, compute-and-forward protocol, treat interference as noise,. . .

How to exploit the broadcast nature of the medium?

Scenario 1 – Communication in a Relay Network

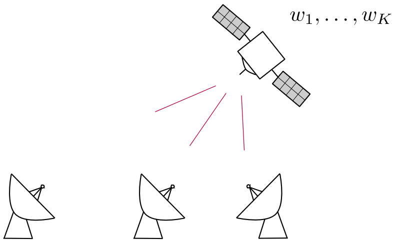


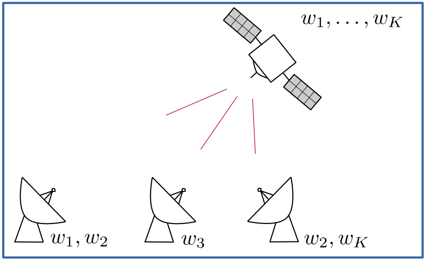


- Base stations transmit to relay.
- Some users can overhear.

- How should the relay encode the messages to efficiently exploit side information?

Scenario 2 – Retransmission in a Broadcast Channel





- Satellite broadcasts multiple packets to receivers
- Because of varying channel conditions, each receiver decodes only a subset of all packets

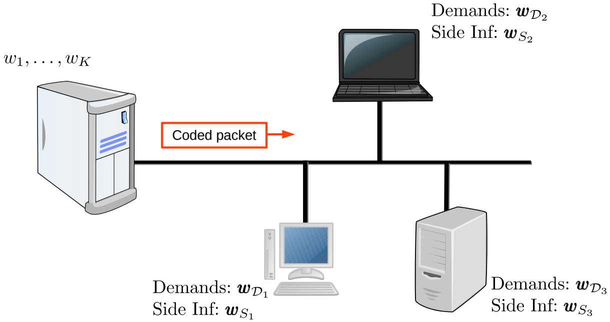
- How should the satellite re-encode the messages to use receiver side information?

Other applications: physical-layer network coding, multi-way relay channels, receivers with cache memory, sensor networks, etc.

Index Coding (at network layer)

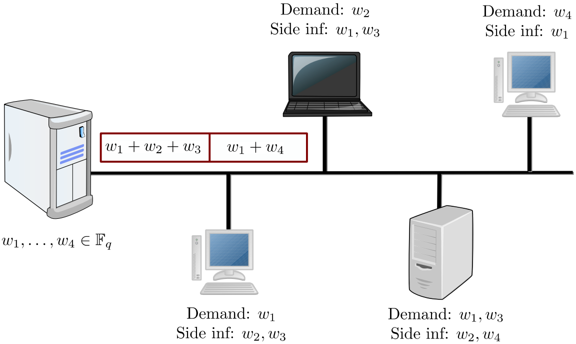
Bar-Yossef, Birk, Jayram & Kol T-IT Mar'11

Rx$_i$ knows S_i and demands $\mathcal{D}_i$, where $S_i, \mathcal{D}_i \subset \{1, \ldots, K\}$.



- **Objective**: Minimize the packet length.
- **Assumption**: Channel coding (performed separately) provides a noise-free channel.

Index Coding (at network layer)

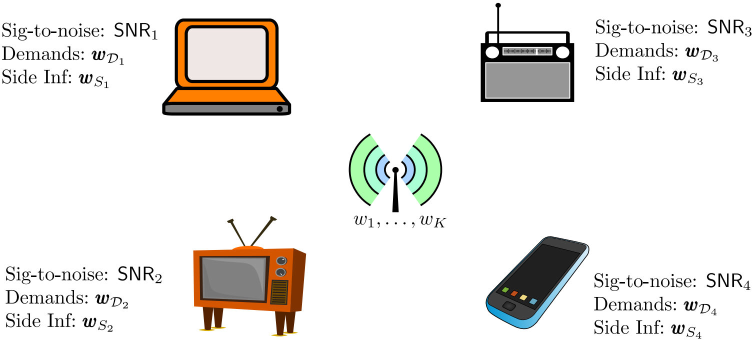


Index coding is..

- representative of all wireline network coding problems (El Rouayheb et al. `T-IT Jul'10`, Effros et al. `T-IT Mar'15`)
- related to coding for distributed storage (Shanmugam & Dimakis, Mazumdar `ISIT'14`)

But, separating index coding from channel coding is not optimal

Index Coding (at physical layer)



Advantage:

- Rx side information can also be used for channel coding/decoding

Objective:

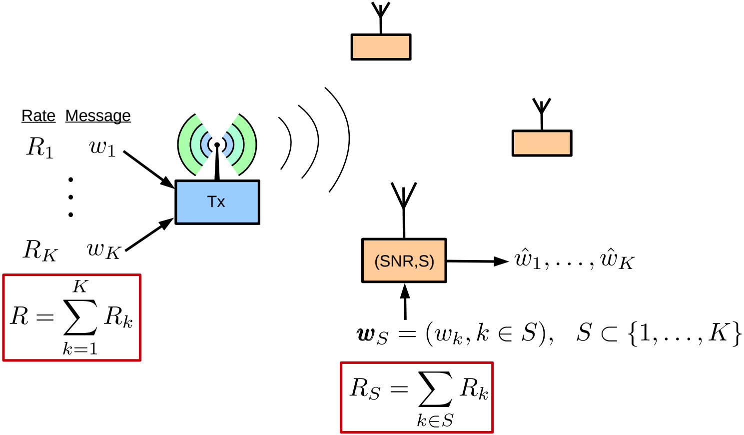
- Maximize achievable rates $R_1, \ldots, R_K$ of the K messages
- Minimize SNR requirements

Literature

- Capacity region (Gaussian broadcast)
  - 2 receivers, general message demands:
    Wu ISIT'07, Xue & Sandhu ITW'07
  - 3 receivers, private messages:
    Sima & Chen ISIT'14,
    Asadi et al. T-IT, Oct'15
  - Every receiver demands all messages:
    Tuncel IT Apr'06, Xie CWIT'07, Kramer & Shamai ITW'07,
    Oechtering et al. IT Jan'08
- Codes for noisy discrete memoryless broadcast channels
  - Use side information to reduce the length of error-correcting codes:
    Dau et al. IT Mar'13, Thomas & Rajan ISIT'15
  - More error-correction capability for larger side information:
    van Dijk et al. ISIT'01, Xiao et al. CISS'06, Kelly & Kliewer ISIT'10, Luo et al. T-IT Mar'10, Barbosa & Costa ITW'11, Ma et al. PIMRC'12

This tutorial: Gaussian broadcast, every receiver demands all messages.

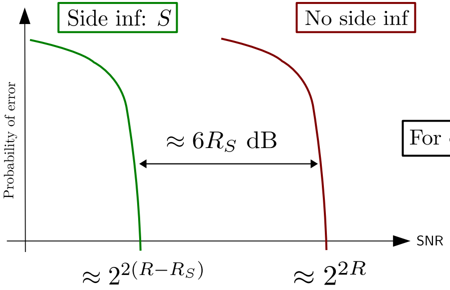
Capacity region



Theorem (Tuncel, `IT Apr'06`)
$(R_1, \ldots, R_K)$ is achievable iff for every receiver (SNR, S)

$$^1/_2 \log_2 (1 + \text{SNR}) > R - R_S.$$

Properties of good codes



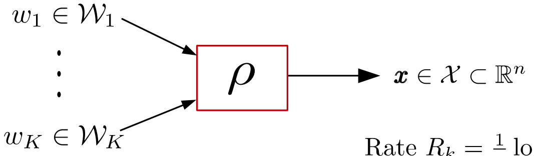
For every $S \subset \{1, \ldots, K\}$

1 b/dim of side inf $\Rightarrow$ 6 dB additional coding gain.

- A good code should be:
  1. good in single-user AWGN channel $\Rightarrow$ for receivers with $S = \varnothing$.
  2. efficient in converting side information into additional coding gain.

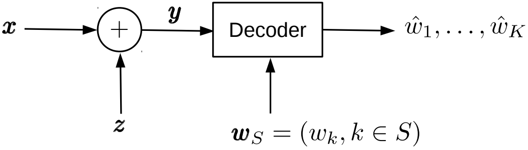
Index Coding for Gaussian Broadcast Channel

Transmitter



Rate $R_k = \frac{1}{n} \log_2 |\mathcal{W}_k|$

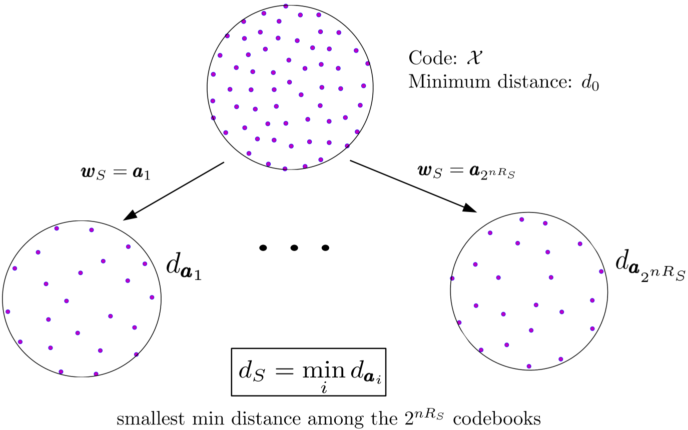
A receiver (SNR, S)



Variance $\frac{1}{\mathsf{SNR}} \cdot \frac{\mathbb{E} \|\boldsymbol{x}\|^2}{n}$

Side information rate $R_S = \sum_{k \in S} R_k$

Decoding



- Additional squared distance gain $= 10\log_{10}\left(d_S^2/d_0^2\right)$ dB.
- Normalized gain $= \frac{1}{R_S}\,10\log_{10}\left(d_S^2/d_0^2\right)$ dB/b/dim.

Measuring the efficiency

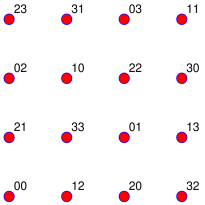
Definition

The *side information gain* of the index code $(\rho, \mathcal{X})$ is

$$\Gamma(\mathcal{X}) = \min_{\varnothing \subsetneq S \subsetneq \{1,\ldots,K\}} \frac{10\log_{10}\left(d_S^2/d_0^2\right)}{R_S} \text{ dB/b/dim.}$$

- Side information of rate $R_S \Rightarrow$
  approx. SNR gain of at least $\Gamma \times R_S$ dB.
- A good index code has
  1. a large d_0: good performance with $S = \varnothing$,
  2. a large $\Gamma(\mathcal{X})$: maximize the gain from side information.

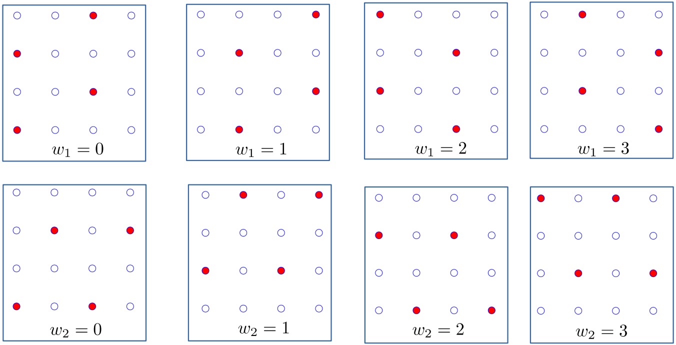
Example – 16-QAM, $K = 2$, $\mathcal{W}_1 = \mathcal{W}_2 = \{0, 1, 2, 3\}$



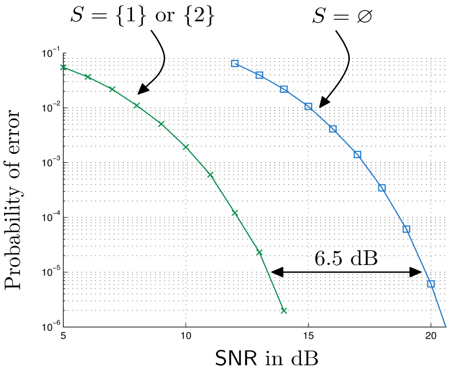
$d_S = 2d_0$ for both $S = \{1\}$ and $S = \{2\}$
$\Gamma \approx 6$ dB/b/dim

$S = \{1\}$

$S = \{2\}$



Performance of 16-QAM index code

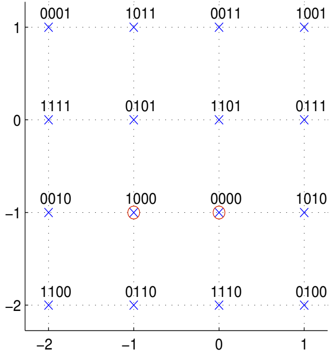


SNR gain over $S = \varnothing$ at $\mathsf{P}_e = 10^{-5}$

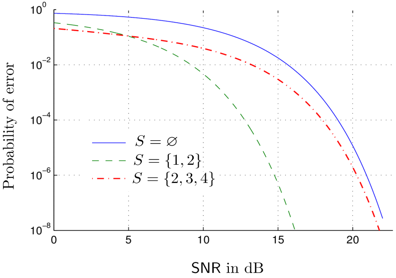
S	Actual gain dB	Predicted $\Gamma \times R_S$ dB
$\{1\}$	6.5	6
$\{2\}$	6.5	6

Example of a bad index code: 16-QAM set-partition

- $K = 4$ messages, $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \{0, 1\}$.



Circles show the subcode with $(w_2, w_3, w_4) = (0, 0, 0)$.

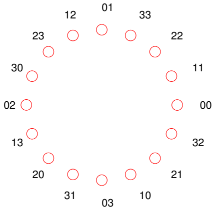


Negligible SNR gain for $S = \{2, 3, 4\}$.

- Set-partition is designed to work only for $S = \{1\}, \{1, 2\}, \{1, 2, 3\}$, not for arbitrary S.

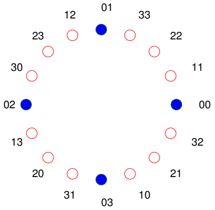
Example: 16-PSK, $K = 2$, $\mathcal{W}_1, \mathcal{W}_2 = \{0, 1, 2, 3\}$

$$n = K = 2,\ R_1 = R_2 = 1 \text{ b/dim.}$$



$S = \varnothing$

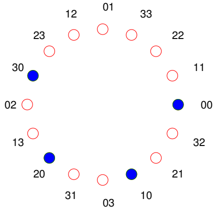
$$d_0 = 2\sin\left(\frac{\pi}{16}\right)$$



$w_1 = 0$

$S = \{1\}$

$$d_S = 2\sin\left(\frac{\pi}{4}\right)$$



$w_2 = 0$

$S = \{2\}$

$$d_S = 2\sin\left(\frac{3\pi}{16}\right)$$

$\Gamma \approx 9.1$ dB/b/dim.

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Index Coding (network layer)

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