On the Optimality of Beamforming for Multi-User MISO Interference Channels with Single-User Detection

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Multi-user MISO Interference Channel (IC) Model

$$Y_i = \sum_{j=1, j\neq i}^{m} \boldsymbol{h}_{ji}^T \boldsymbol{x}_j + N_i$$

- m transmitter-receiver pairs.
- $\boldsymbol{x}_i$: $t_i \times 1$ transmitted signal at transmitter i. $\|\boldsymbol{x}_i\|^2 \leq P_i$.
- Y_i: received scalar signal at receiver i .
- $N_i \sim \mathcal{N}(0,1)$.
- $\boldsymbol{h}_{ji}$: $t_j \times 1$ channel vector from transmitter j to receiver i.
- Transmitter i knows all channel states, receiver i only knows $\boldsymbol{h}_{ii}$.

Single-User Detection Rate Region

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} \mu_i R_i \qquad\qquad\qquad (\mu_i \geq 0) \\
\text{subject to} \quad & R_i = \frac{1}{2}\log\left(1 + \frac{\boldsymbol{h}_{ii}^T \mathbf{S}_i \boldsymbol{h}_{ii}}{1 + \sum_{j=1, j\neq i}^{m} \boldsymbol{h}_{ji}^T \mathbf{S}_j \boldsymbol{h}_{ji}}\right) \\
& \operatorname{tr}(\mathbf{S}_i) \leq P_i, \quad \mathbf{S}_i \succeq \mathbf{0}, \quad i = 1, \cdots m.
\end{aligned}
$$

- $\boldsymbol{x}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{S}_i)$.
- Receiver i decodes $\boldsymbol{x}_i$ from Y_i by treating interference as noise.
- Single-user detection (SUD) rate region is defined as

$$
\bigcup_{\text{all } \boldsymbol{\mu}=[\mu_1,\cdots,\mu_m]} \{R_1, \cdots, R_m\}
$$

Problem Reformulation

- Why: Non-convex optimization problem.

$$\mu_1 \log\left(1+\frac{\mathbf{h}_{11}^T\mathbf{S}_1\mathbf{h}_{11}}{1+\mathbf{h}_{21}^T\mathbf{S}_2\mathbf{h}_{21}}\right)+\mu_2 \log\left(1+\frac{\mathbf{h}_{22}^T\mathbf{S}_2\mathbf{h}_{22}}{1+\mathbf{h}_{12}^T\mathbf{S}_1\mathbf{h}_{12}}\right)$$

- How: non-convex problem $\Rrightarrow$ convex problem

$$\begin{aligned}
\text{maximize} \quad & \mu_1 \log\left(1+\frac{\mathbf{h}_{11}^T\mathbf{S}_1\mathbf{h}_{11}}{1+z_2^2}\right)+\mu_2 \log\left(1+\frac{\mathbf{h}_{22}^T\mathbf{S}_2\mathbf{h}_{22}}{1+z_1^2}\right)\\
\text{subject to} \quad & \mathbf{h}_{12}^T\mathbf{S}_1\mathbf{h}_{12}=z_1^2, \quad \mathbf{h}_{21}^T\mathbf{S}_2\mathbf{h}_{21}=z_2^2,\\
& \operatorname{tr}(\mathbf{S}_i)\le P_i, \quad \mathbf{S}_i\succeq 0, \quad i=1,2.
\end{aligned}$$

New Convex Optimization Problem

$$
\begin{aligned}
\max \quad & \boldsymbol{h}_{ii}^T \mathbf{S}_i \boldsymbol{h}_{ii} \\
\text{subject to} \quad & \boldsymbol{h}_{ij}^T \mathbf{S}_i \boldsymbol{h}_{ij} \leq z_{ij}^2, \\
& \mathrm{tr}(\mathbf{S}_i) \leq P_i, \quad \mathbf{S}_i \succeq 0 \\
& i, j = 1, \ldots, m, \quad i \neq j,
\end{aligned}
$$

- Preselect z_{ji}^2 as the interference power caused by the jth transmitter to the ith receiver.
- Maximize useful signal power with the interference constraint.
- Obtain the optimal covariance matrix for user i.

Comparison of the Complexity

- **Old problem**

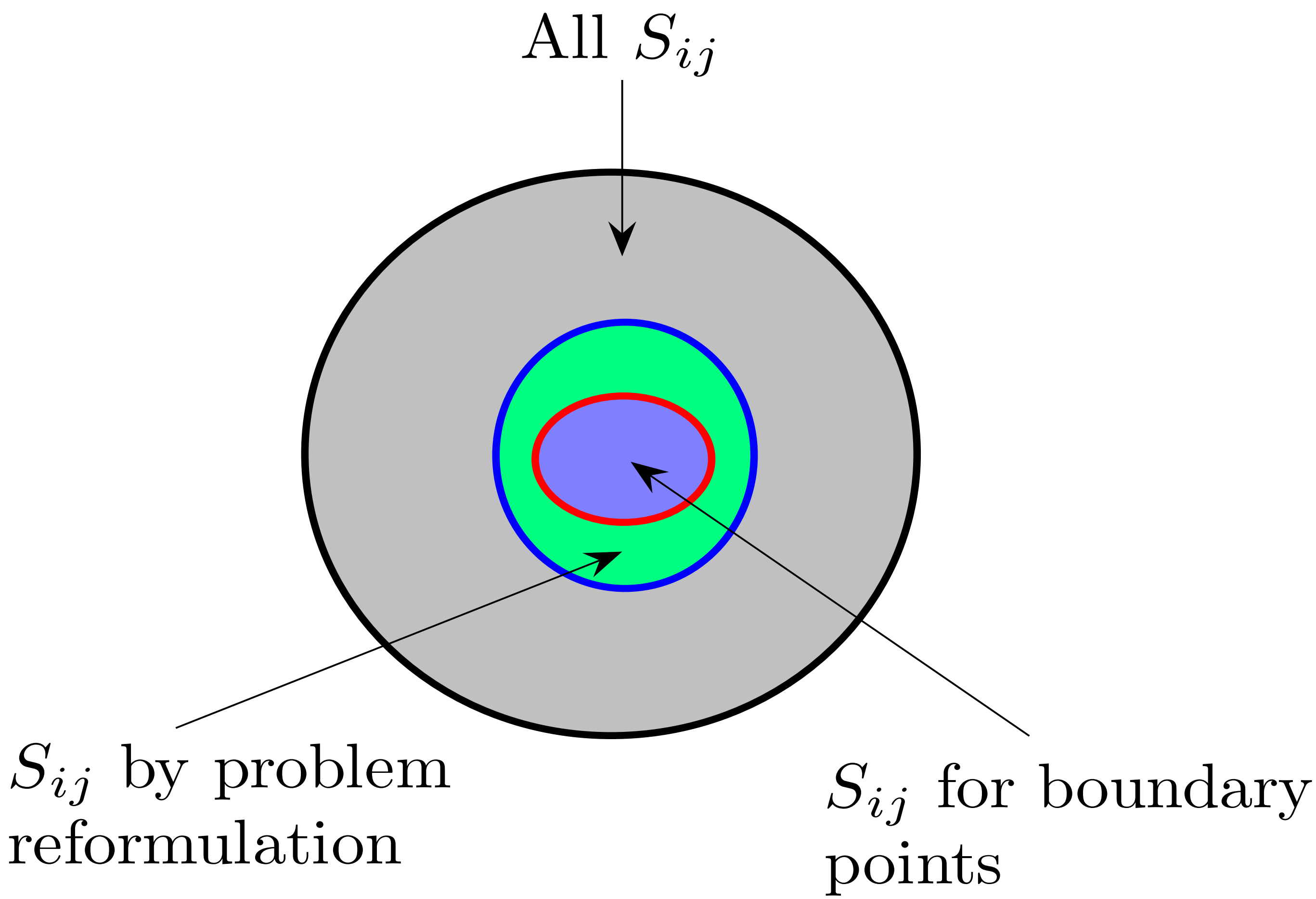
$$\prod_{i=1}^{m} (\text{num of } \mu_i) \;\; \text{non-convex problems}$$

- **New problem**

$$\sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} (\text{num of } z_{ij}) \;\; \text{convex problems}$$

- **Advantage of problem reformulation**
  - Non-convex $\Rrightarrow$ convex
  - Decoupling reduces complexity.

Covariance Matrices



Matrix Optimization Lemma

If

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{21}^T \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \succeq \mathbf{0}, \quad \mathrm{tr}(\mathbf{K}) \leq P$$

and $\mathbf{K}_{11} \succeq \mathbf{0}$ is a preselected sub-matrix, then

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}^T \mathbf{K} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \leq \left(\sqrt{\boldsymbol{x}^T \mathbf{K}_{11} \boldsymbol{x}} + \|\boldsymbol{y}\| \sqrt{P - \mathrm{tr}(\mathbf{K}_{11})} \right)^2$$

and there exists $\mathbf{K}^*$ that achieves the equality with

$$\mathrm{rank}\left(\mathbf{K}^*\right) = \mathrm{rank}\left(\mathbf{K}_{11}\right).$$

Problem Simplification

$$\begin{aligned} \max \quad & \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}^T \mathbf{K} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \\ \text{subject to} \quad & h_i(\mathbf{K}_{11}) = 0, \quad g_j(\mathbf{K}_{11}) \leq 0 \\ & \mathrm{tr}(\mathbf{K}) \leq P, \quad \mathbf{K} \succeq \mathbf{0}, \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \max \quad & \left(\sqrt{\boldsymbol{x}^T \mathbf{K}_{11} \boldsymbol{x}} + \|\boldsymbol{y}\| \sqrt{P - \mathrm{tr}(\mathbf{K}_{11})} \right)^2 \\ \text{subject to} \quad & h_i(\mathbf{K}_{11}) = 0, \quad g_j(\mathbf{K}_{11}) \leq 0, \\ & \mathrm{tr}(\mathbf{K}_{11}) \leq P, \quad \mathbf{K} \succeq \mathbf{0}. \end{aligned}$$

- Reduce the dimension of the matrix and vectors.

A Revisit of the Problem

$$\begin{aligned} \max \quad & \boldsymbol{h}_{mm}^T \mathbf{S}_m \boldsymbol{h}_{mm} \\ \text{subject to} \quad & \boldsymbol{h}_{mj}^T \mathbf{S}_i \boldsymbol{h}_{mj} \le z_{mj}^2, \quad j = 1, \ldots, m-1 \\ & \mathrm{tr}(\mathbf{S}_m) \le P_m, \quad \mathbf{S}_m \succeq 0. \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \max \quad & \begin{bmatrix} \boldsymbol{h} \\ \widehat{\boldsymbol{h}} \end{bmatrix}^T \tilde{\mathbf{S}} \begin{bmatrix} \boldsymbol{h} \\ \widehat{\boldsymbol{h}} \end{bmatrix} \\ \text{subject to} \quad & \begin{bmatrix} \boldsymbol{h}_j \\ \mathbf{0} \end{bmatrix}^T \tilde{\mathbf{S}} \begin{bmatrix} \boldsymbol{h}_j \\ \mathbf{0} \end{bmatrix} \le z_{mj}^2, \quad j = 1, \cdots m-1, \\ & \mathrm{tr}\left(\tilde{\mathbf{S}}\right) \le P_m, \quad \tilde{\mathbf{S}} \succeq \mathbf{0}, \end{aligned}$$

A Revisit of the Problem (continue)

$\Rightarrow$

$$\begin{aligned} \max \quad & \boldsymbol{h}^T \tilde{\mathbf{S}}_{11} \boldsymbol{h} \\ \text{subject to} \quad & \boldsymbol{h}_j^T \tilde{\mathbf{S}}_{11} \boldsymbol{h}_j \le z_{mj}^2, \quad j = 1, \ldots, m-1 \\ & \mathrm{tr}(\tilde{\mathbf{S}}_{11}) \le P_m, \quad \tilde{\mathbf{S}}_{11} \succeq 0, \end{aligned}$$

where

$$\dim(\boldsymbol{h}) = \dim(\boldsymbol{h}_j) = \min\{t_m, m-1\} \triangleq \bar{m}$$

$$\tilde{\mathbf{S}} = \begin{bmatrix} \tilde{\mathbf{S}}_{11} & \tilde{\mathbf{S}}_{21}^T \\ \tilde{\mathbf{S}}_{21} & \tilde{\mathbf{S}}_{22} \end{bmatrix}.$$

Problem Simplification Procedure

- **Step 1:**
  - Singular value decomposition

$$\boldsymbol{h}_{m1} = \mathbf{U}_1 \begin{bmatrix} \|\boldsymbol{h}_{m1}\| \\ \mathbf{0}_{(t_m-1)\times 1} \end{bmatrix}, \quad \text{where } \mathbf{U}_1^T\mathbf{U}_1 = \mathbf{I}$$

  - then update

$$\mathbf{h}_{mj}^{(1)} = \mathbf{U}_1^T\mathbf{h}_{mj}, \quad j = 1, \cdots, m.$$

$$\mathbf{S}_m^{(1)} = \mathbf{U}_1^T\mathbf{S}_m\mathbf{U}_1.$$

Problem Simplification Procedure (continue)

- **Step 2:**

$$\mathbf{h}_{m2}^{(1)} = \begin{bmatrix} \left(\mathbf{h}_{m2}^{(1)}\right)_1 \\ \left(\mathbf{h}_{m2}^{(1)}\right)_{2,\cdots,t_m} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \left(\mathbf{h}_{m2}^{(1)}\right)_1 \\ \left\| \left(\mathbf{h}_{m2}^{(1)}\right)_{2,\cdots,t_m} \right\| \\ \mathbf{0}_{(t_m-2)\times 1} \end{bmatrix}, \quad \mathbf{U}_2^T \mathbf{U}_2 = \mathbf{I}$$

$$\mathbf{h}_{mj}^{(2)} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix}^T \mathbf{h}_{mj}^{(1)}, \quad j = 1, \cdots m.$$

$$\mathbf{S}_m^{(2)} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix}^T \mathbf{S}_m^{(1)} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2 \end{bmatrix}$$

Final State

$$\max \quad \begin{bmatrix} \boldsymbol{h} \\ \widehat{\boldsymbol{h}} \end{bmatrix}^T \begin{bmatrix} \tilde{\mathbf{S}}_{11} & \tilde{\mathbf{S}}_{21}^T \\ \tilde{\mathbf{S}}_{21} & \tilde{\mathbf{S}}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{h} \\ \widehat{\boldsymbol{h}} \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} \boldsymbol{h}_j \\ \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \tilde{\mathbf{S}}_{11} & \tilde{\mathbf{S}}_{21}^T \\ \tilde{\mathbf{S}}_{21} & \tilde{\mathbf{S}}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{h}_j \\ \mathbf{0} \end{bmatrix} \leq z_{mj}^2, \quad j = 1, \cdots \bar{m},$$

$$\mathrm{tr}\left(\begin{bmatrix} \tilde{\mathbf{S}}_{11} & \tilde{\mathbf{S}}_{21}^T \\ \tilde{\mathbf{S}}_{21} & \tilde{\mathbf{S}}_{22} \end{bmatrix} \right) \leq P_m, \quad \begin{bmatrix} \tilde{\mathbf{S}}_{11} & \tilde{\mathbf{S}}_{21}^T \\ \tilde{\mathbf{S}}_{21} & \tilde{\mathbf{S}}_{22} \end{bmatrix} \succeq \mathbf{0},$$

Optimality of Beamforming

- **Theorem:**

  There exist an $\mathbf{S}^* \succeq \mathbf{0}$ and $\text{rank}(\mathbf{S}^*) \leq 1$ that is optimal for

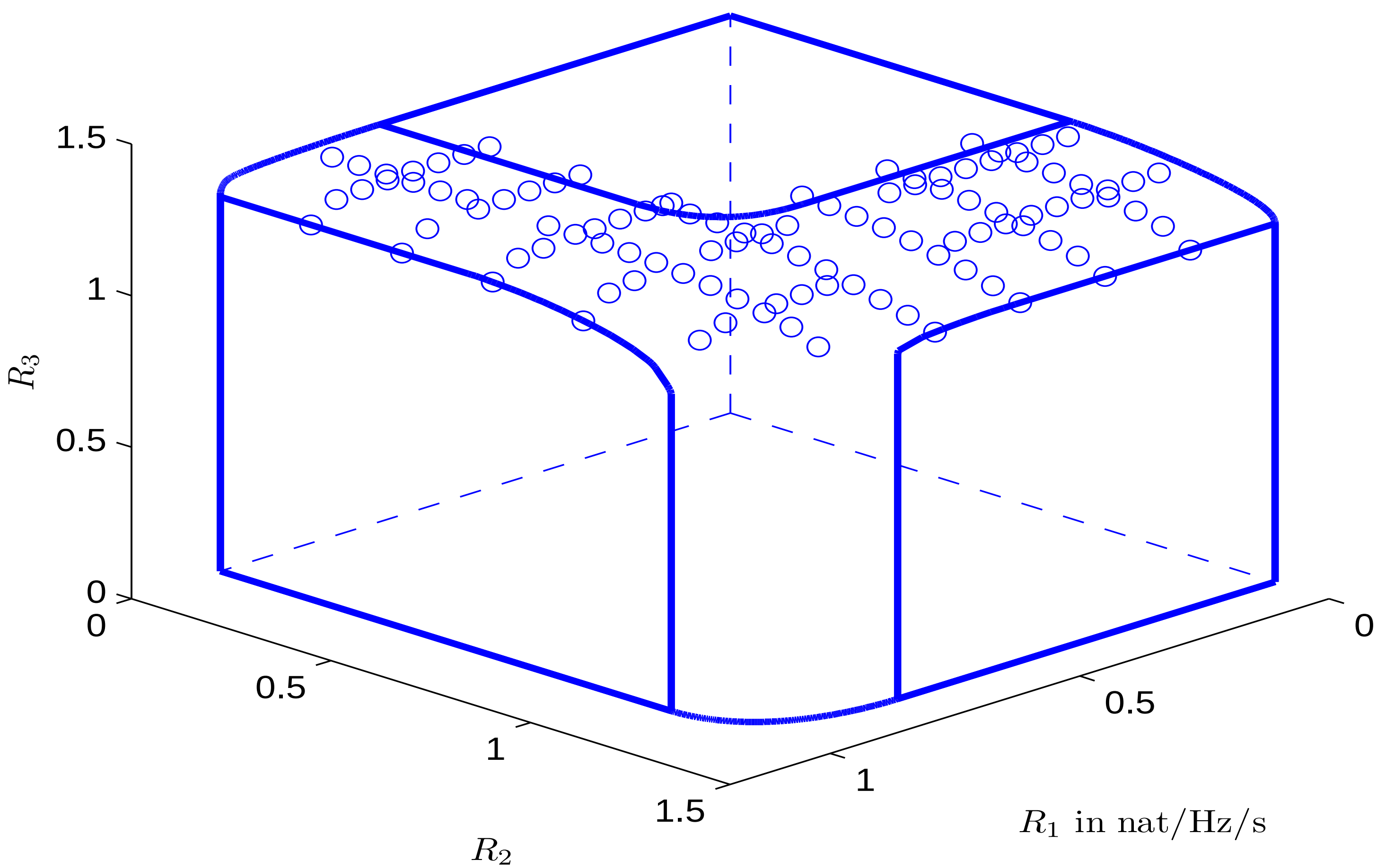
$$\begin{aligned} \max \quad & \boldsymbol{h}_{mm}^T \mathbf{S}_m \boldsymbol{h}_{mm} \\ \text{subject to} \quad & \boldsymbol{h}_{mj}^T \mathbf{S}_i \boldsymbol{h}_{mj} \leq z_{mj}^2, \quad j = 1, \ldots, m-1 \\ & \text{tr}(\mathbf{S}_m) \leq P_m, \quad \mathbf{S}_m \succeq 0. \end{aligned}$$

- **Proof:**
  - Karush-Kuhn-Tucker conditions.
  - Extension of Sylvester's Law of Inertia.

Conclusion

For an m-user MISO interference, the boundary points of the single-user detection rate region can be achieved by restricting each transmitter to implementing beamforming.

Numerical Example



The convex hull of the SUD rate regions of a three-user MISO IC.