Impact of Noise on the Nonlinear Fourier Transform

Sander Wahls

Delft Center for Systems and Control, TU Delft

`http://www.dcsc.tudelft.nl/~swahls`

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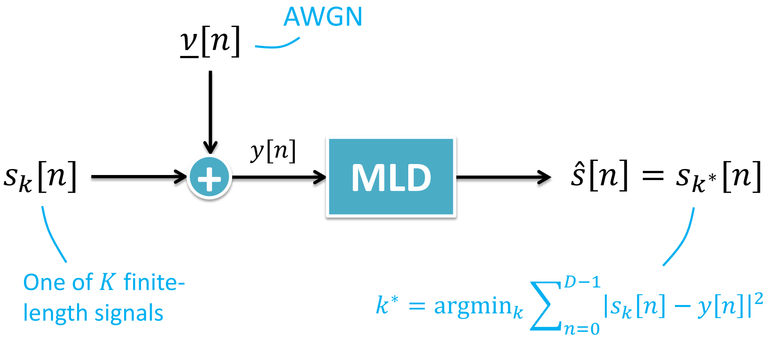
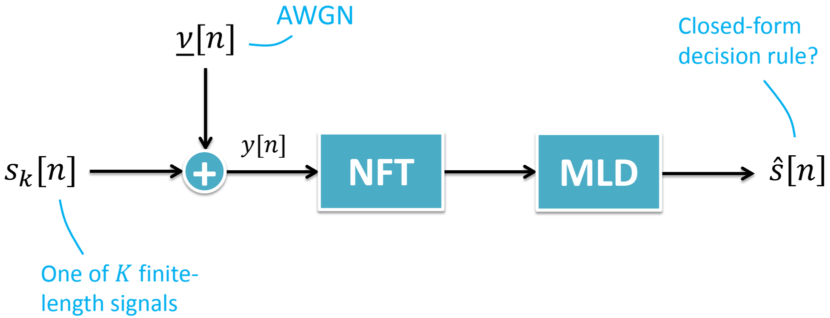
Motivation



$\underline{v}[n]$
AWGN
$s_k[n]$
$y[n]$
MLD
$\hat{s}[n] = s_{k^*}[n]$
One of K finite-length signals
$k^* = \text{argmin}_k \sum_{n=0}^{D-1} |s_k[n] - y[n]|^2$

Motivation

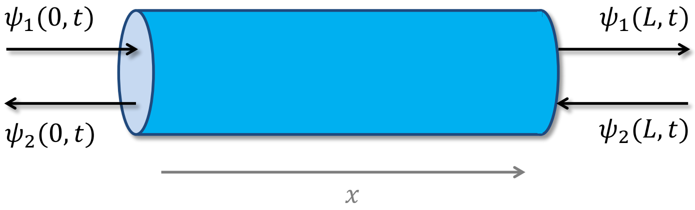


(Thanks to Vahid Aref, who brought this question to my attention.)

Outline

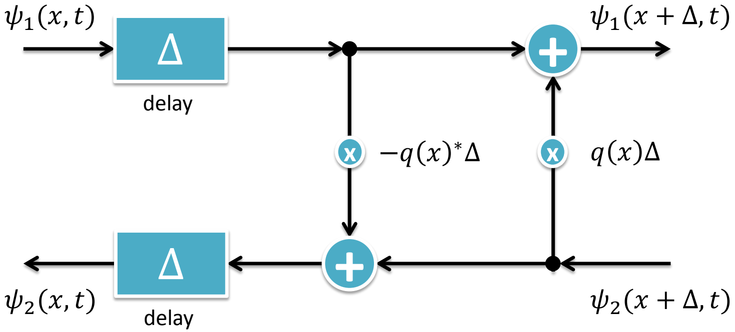
Prelude: A Scattering Medium

A scattering medium



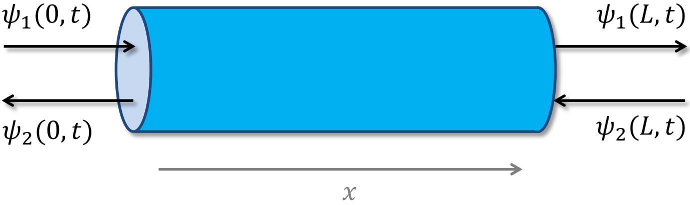
- Two waves interact in a scattering medium: $\psi_1(x,t)$ and $\psi_2(x,t)$, where x denotes location and t time
- ψ_1 travels to the right, ψ_2 to the left

Interaction over an infinitesimal amount of space



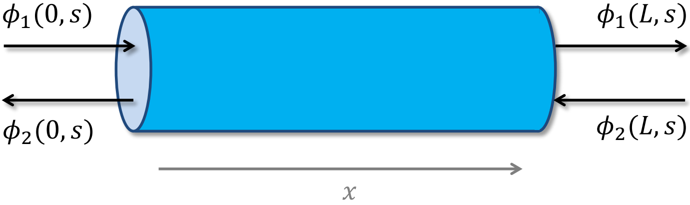
The profile $q(x)$ describes the medium [Bruckstein et al., 1985].

Evolution of the waves in the time domain



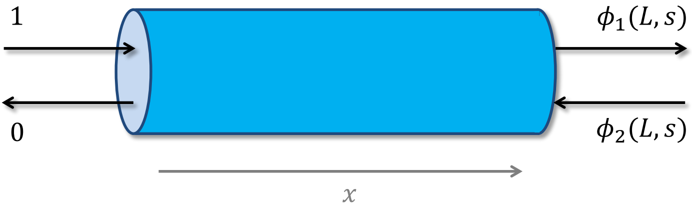
$$\frac{d\boldsymbol{\psi}}{dx} = \begin{bmatrix} -\frac{\partial}{\partial t} & q(x) \\ -q^*(x) & \frac{\partial}{\partial t} \end{bmatrix} \boldsymbol{\psi}, \qquad \boldsymbol{\psi}(x,t) = \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix}$$

Evolution of the waves in the Laplace domain



$$\frac{d\boldsymbol{\phi}}{dx} = \begin{bmatrix} -s & q(x) \\ -q^*(x) & s \end{bmatrix} \boldsymbol{\phi}, \qquad \boldsymbol{\phi}(x,s) = Laplace\{\boldsymbol{\psi}(x,t)\}$$

Probing the medium

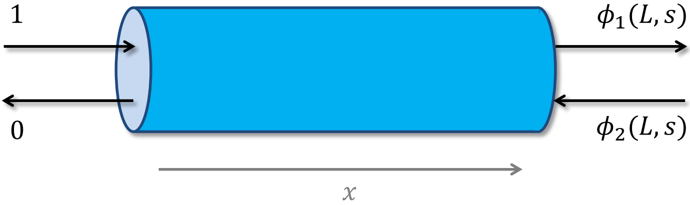


We probe the medium with the inputs

$$\phi_1(0,s) = 1 \quad \text{and} \quad \phi_2(L,s),$$

where the latter is specified indirectly through $\phi_2(0,s) = 0$.

Reconstruction of the medium



The profile $q(x)$ can be reconstructed from the scattering data:

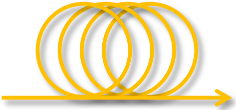
1. the reflection coefficient ϕ_2/ϕ_1 for $s = j\omega$,
2. the unstable poles of the reflection coefficient, and
3. the residuals of the reflection coefficient at these poles,

all taken at the right end of the medium, $x = L$.

Fiber-Optic Communication Using NFTs

Channel model

$u(\chi = 0, \tau)$



$u(\chi_1, \tau)$

Normalized nonlinear Schrödinger equation:

$$j\frac{\partial u}{\partial \chi} = \frac{\partial^2 u}{\partial \tau^2} + 2|u|^2 u, \qquad u = u(\chi, \tau)$$

- Models ideal, loss- and noise-free single-mode fiber
- u = complex envelope, χ = location, τ = retarded time

Nonlinear Fourier transform (of a vanishing signal)

Consider the scattering problem

$$\frac{d\boldsymbol{\phi}}{d\tau} = \begin{bmatrix} -s & u(\chi_0,\tau) \\ u^*(\chi_0,\tau) & s \end{bmatrix} \boldsymbol{\phi}, \quad \lim_{\tau\to-\infty} \boldsymbol{\phi}(s,\tau) = \begin{bmatrix} e^{-s\tau} \\ 0 \end{bmatrix}$$

We normalize the wave functions as

$$\alpha(s) := \lim_{\tau\to\infty} e^{s\tau}\phi_1(\tau,s), \qquad \beta(s) := \lim_{\tau\to\infty} e^{-s\tau}\phi_2(\tau,s)$$

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Nonlinear Fourier transform of $u(\tau) = u(\chi_0,\tau)$:

$$\hat{u}(j\omega) := \frac{\beta(j\omega)}{\alpha(j\omega)}; \quad \alpha(s_k) = 0,\ \Re(s_k) > 0; \quad \tilde{u}_k := \beta(s_k) \Big/ \frac{d\alpha}{ds}\Big|_{s=s_k}$$

This is essentially the scattering data of an imaginary scattering medium with profile $q(x) = u(\chi_0,\tau)|_{\tau=x}$.

Fiber-optic communication using NFTs

$u(0, \tau)$

$\uparrow$

Inverse NFT

$|$

s_k

$\tilde{u}_k(0)$

$\hat{u}(0, j\omega)$

TX

Fiber-optic communication using NFTs

Ideal Fiber

$u(0,\tau)$ $\longrightarrow$ $u(\chi_1, t)$

$j\frac{\partial u}{\partial \chi} = \frac{\partial^2 u}{\partial \tau^2} + 2u|u|^2$

$\uparrow$

Inverse NFT

$|$

s_k

$\tilde{u}_k(0)$

$\hat{u}(0, j\omega)$

TX

Fiber-optic communication using NFTs

Ideal Fiber

$u(0,\tau) \xrightarrow[j\frac{\partial u}{\partial \chi} = \frac{\partial^2 u}{\partial \tau^2} + 2u|u|^2]{} u(\chi_1, t)$

TX	RX
$\uparrow$ Inverse NFT $\vert$	$\vert$ Forward NFT $\downarrow$
s_k	s_k
$\tilde{u}_k(0)$	$\tilde{u}_k(\chi_1)$
$\hat{u}(0, j\omega)$	$\hat{u}(\chi_1, j\omega)$
TX	**RX**

Fiber-optic communication using NFTs

Ideal Fiber

$u(0,\tau)$ $\xrightarrow[j\frac{\partial u}{\partial \chi} = \frac{\partial^2 u}{\partial \tau^2} + 2u|u|^2]{}$ $u(\chi_1, t)$

$\uparrow$ Inverse NFT $|$ $\qquad$ $|$ Forward NFT $\downarrow$

TX	RX		
s_k	s_k		s_k
$\tilde{u}_k(0)$	$\tilde{u}_k(\chi_1)$	$=$	$\tilde{u}_k(0)e^{-4s_k^2\chi_1}$
$\hat{u}(0, j\omega)$	$\hat{u}(\chi_1, j\omega)$		$\hat{u}(0, j\omega)e^{-4j\omega^2\chi_1}$

TX $\qquad$ **RX**

Fiber-optic communication using NFTs

Ideal Fiber

$u(0,\tau)$ $\xrightarrow[j\frac{\partial u}{\partial \chi} = \frac{\partial^2 u}{\partial \tau^2} + 2u|u|^2]{}$ $u(\chi_1, t)$

$\uparrow$ Inverse NFT | | Forward NFT $\downarrow$

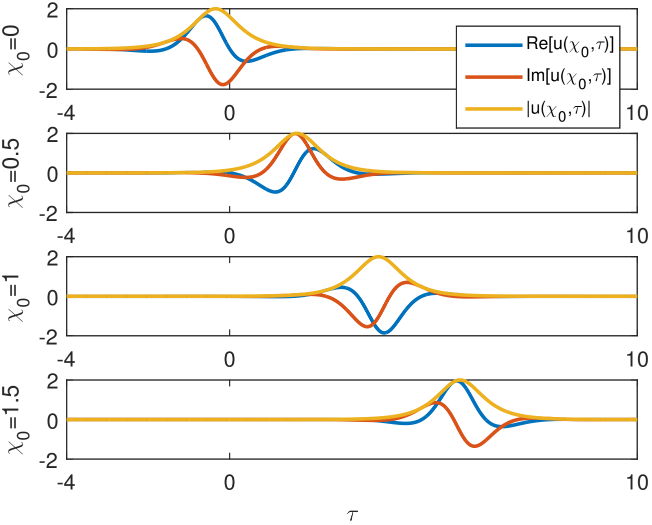
s_k | s_k | | s_k
$\tilde{u}_k(0)$ | $\tilde{u}_k(\chi_1)$ | $=$ | $\tilde{u}_k(0)e^{-4s_k^2\chi_1}$
$\hat{u}(0,j\omega)$ | $\hat{u}(\chi_1,j\omega)$ | | $\hat{u}(0,j\omega)e^{-4j\omega^2\chi_1}$

TX **RX**

Problem: The impact of noise on the NFT is difficult to assess.

Some Known Results

The soliton



Re[u(χ_0,τ)]
Im[u(χ_0,τ)]
|u(χ_0,τ)|
χ_0=0
χ_0=0.5
χ_0=1
χ_0=1.5
2
0
-2
-4
0
10
τ

The soliton

The soliton

$$u(\chi,\tau) = 2j\eta e^{-j(2\xi\tau - 4(\xi^2-\eta^2)\chi + \psi)}\text{sech}(2\eta\tau - 8\xi\eta\chi - \delta)$$

is determined by the parameters

- $\eta \to$ amplitude, phase, time-scale and velocity
- $\xi \to$ phase and velocity
- $\psi \to$ phase-shift
- $\delta \to$ time-shift

The soliton

The soliton

$$u(\chi,\tau) = 2j\eta e^{-j(2\xi\tau - 4(\xi^2-\eta^2)\chi+\psi)}\text{sech}(2\eta\tau - 8\xi\eta\chi - \delta)$$

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- $\eta \rightarrow$ amplitude, phase, time-scale and velocity
- $\xi \rightarrow$ phase and velocity
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- $\delta \rightarrow$ time-shift

The NFT of $u(\chi_0,\tau)$ is $\hat{q}(\chi_0, j\omega) = 0$, s_1, $\tilde{q}_1(\chi_0)$ and satistifies

$$\xi = \Re\{s_1\}, \quad \eta = \Im\{s_1\}, \quad e^{\delta} = \frac{|\tilde{q}_1(0)|}{2\eta}, \quad e^{i\psi} = \frac{\tilde{q}_1(0)}{|\tilde{q}_1(0)|}$$

The soliton

The evolution of a soliton in non-ideal fiber is quite well-studied.

Two classic results:

- *Kaup (1976):* Decay of the amplitude η in lossy fiber.
  "A perturbation expansion for the Zakharov-Shabat inverse scattering transform," SIAM J. Appl. Math. 31(1)

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The main tool in these derivations is perturbation theory.

See the book of Hasegawa (1995) for an overview.

(Some) recent work

Much recent work on noise modeling for more general types of signals in works on capacity estimates. Some examples:

- Meron, Shtaif & Feder (arXiv, 2012)
- Zhang & Chen (ISIT 2015)
- Shevchenko et al. (ITW, 2015)
- Kazakopoulos & Moustakas (ISIT 2016)
- Derevyanko, Prilepsky & Turitsyn (Nature Commun., 2016)

All of these works rely on perturbation theory.

Issues: High SNR only, restricted signal sets (e.g., multisolitons), often more assumptions (well-separated solitons, short fiber, ...)

A New Result

Why we need another noise model for the NFT

Current results are <u>all</u> based on perturbation theory

$\Rightarrow$ Only valid for high SNR

- The high SNR regime is interesting for capacity estimates
- But it is not relevant for signal detection
- Another approach is needed!

Discrete-time NFT of a noisy signal

We consider noisy samples of a deterministic signal,

$$\underline{\mathrm{q}}[n] := q(t_n) + \underline{\nu}[n], \qquad n = 0, 1, \ldots, D-1,$$

where $\underline{\nu}[n]$ is i.i.d. circular symmetric white Gaussian noise

Discretization of the scattering problem:

$$\underline{\boldsymbol{\phi}}[n+1] = \begin{bmatrix} 1 & \epsilon \underline{\mathrm{q}}[n] z^{-1} \\ -\epsilon \underline{\mathrm{q}}^*[n] & z^{-1} \end{bmatrix} \underline{\boldsymbol{\phi}}[n], \qquad z = e^{2s\epsilon}$$

We will investigate the first two moments of $\underline{\boldsymbol{\phi}}[D]$, from which the discrete-time NFT is defined as in the c-t case (skipped)

Real-augmented model I

We now rewrite the iteration for the discrete wave-vector:

$$\begin{aligned}
\underline{\phi}[n+1] &= \begin{bmatrix} 1 & \epsilon \underline{\mathrm{q}}[n]z^{-1} \\ -\epsilon \underline{\mathrm{q}}^*[n] & z^{-1} \end{bmatrix} \underline{\phi}[n] \\
&= \begin{bmatrix} 1 & \epsilon q(t_n)z^{-1} + \epsilon \underline{\nu}[n]z^{-1} \\ -\epsilon q^*(t_n) - \epsilon \underline{\nu}^*[n] & z^{-1} \end{bmatrix} \underline{\phi}[n] \\
&= (\boldsymbol{A}^c[n] + \underline{\nu}_r[n]\boldsymbol{L}^c + \underline{\nu}_i[n]\boldsymbol{M}^c)\underline{\phi}[n],
\end{aligned}$$

where

$$\boldsymbol{A}^c[n] := \begin{bmatrix} 1 & \epsilon q[n]z^{-1} \\ -\epsilon q^*[n] & z^{-1} \end{bmatrix}, \quad \boldsymbol{L}^c := \epsilon \begin{bmatrix} 0 & z^{-1} \\ -1 & 0 \end{bmatrix},$$

$$\boldsymbol{M}^c := j\epsilon \begin{bmatrix} 0 & z^{-1} \\ 1 & 0 \end{bmatrix},$$

and $\underline{\nu}_r[n] := \Re \underline{\nu}[n]$, $\underline{\nu}_i[n] := \Im \underline{\nu}[n]$.

Real-augmented model II

Now, we separate the real and imaginary parts into the vector

$$\underline{\mathbf{x}}[n] := \begin{bmatrix} \Re \underline{\boldsymbol{\phi}}[n] \\ \Im \underline{\boldsymbol{\phi}}[n] \end{bmatrix}.$$

The update for $\underline{\mathbf{x}}[n]$ is

$$\boxed{\underline{\boldsymbol{x}}[n+1] = (\boldsymbol{A}[n] + \underline{\nu}_r[n]\boldsymbol{L} + \underline{\nu}_i[n]\boldsymbol{M})\underline{\boldsymbol{x}}[n],}$$

where

$$\boldsymbol{A}[n] := \mathrm{c2r}\{\boldsymbol{A}^c[n]\}, \qquad \boldsymbol{L} := \mathrm{c2r}\{\boldsymbol{L}^c\}, \qquad \boldsymbol{M} := \mathrm{c2r}\{\boldsymbol{M}^c\},$$

and

$$\mathrm{c2r}\{\boldsymbol{X}\} := \begin{bmatrix} \Re \boldsymbol{X} & -\Im \boldsymbol{X} \\ \Im \boldsymbol{X} & \Re \boldsymbol{X} \end{bmatrix}.$$

Mean and variance of the real-augmented model

Wang and Balakrishnan (2002): Let $\boldsymbol{\mu}[n] := \mathbb{E}\{\underline{\mathbf{x}}[n]\}$ denote the mean and $\boldsymbol{V}[n] := \mathbb{E}\{\underline{\boldsymbol{x}}[n]\underline{\boldsymbol{x}}[n]^T\}$ the variance of the real-augmented wave-vector, respectively. Then,

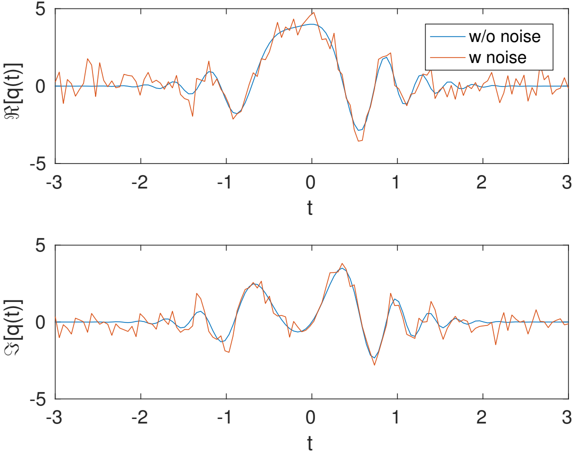
$$\boldsymbol{\mu}[n+1] = \boldsymbol{A}[n]\boldsymbol{\mu}[n],$$

$$\boldsymbol{V}[n+1] = \boldsymbol{A}[n]\boldsymbol{V}[n]\boldsymbol{A}[n]^T + \frac{\sigma^2}{2}(\boldsymbol{L}\boldsymbol{V}[n]\boldsymbol{L}^T + \boldsymbol{M}\boldsymbol{V}[n]\boldsymbol{M}^T),$$

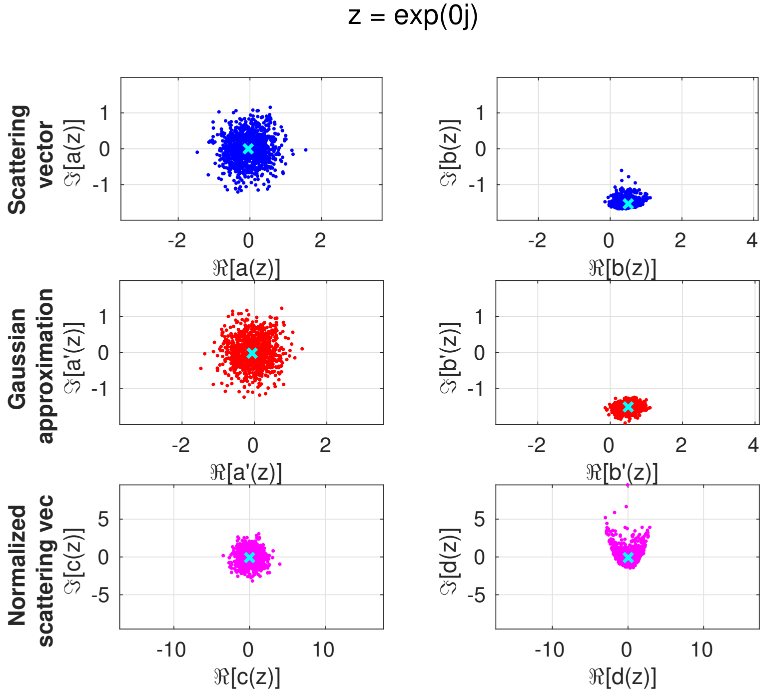
where $\sigma^2 = \mathbb{E}[|\nu[n]|^2]$ is the variance of the noise.

- This result is <u>exact for any noise level</u>!
- Actual distribution? Spatial evolution?

Numerical example



$$u(\tau) = 4\exp(-\tau^2 + 6j\tau^2 + 2j\tau), \qquad \sigma^2 = 0.7^2$$


z = exp(0j)
Scattering vector
ℑ[a(z)]
ℜ[a(z)]
ℑ[b(z)]
ℜ[b(z)]
Gaussian approximation
ℑ[a'(z)]
ℜ[a'(z)]
ℑ[b'(z)]
ℜ[b'(z)]
Normalized scattering vec
ℑ[c(z)]
ℜ[c(z)]
ℑ[d(z)]
ℜ[d(z)]

Conclusion

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The End